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RADC-TDR-63-147
ASTIA Document No. AD-

FINAL REPORT

INVESTIGATION OF ANALOG AND DIGITAL COMMUNICATION SYSTEMS (PHASE 3 REPORT)

TECHNICAL DOCUMENTARY REPORT RADC-TDR-63-147

May 1963

Research and Technology Division
Rome Air Development Center
Air Force Systems Command
United States Air Force
Griffiss Air Force Base, New York

Project No. 4519, Task No. 451903

(Prepared under Contract No. AF 30(602)-2210
by Cornell Aeronautical Laboratory, Inc.
Buffalo 21, New York
Author: John G. Lawton)

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CAL Report No. UA-1420-S-3

FOREWORD

The work described in this report was accomplished by the Cornell Aeronautical Laboratory, Buffalo, New York, for the Rome Air Development Center, Griffiss Air Force Base, New York, during the period February 1962 through February 1963, and has been designated "Phase 3" under Contract No. AF 30(602)-2210, Project No. 4519, Task No. 451903. The results of the Phase 1 and 2 studies have been previously presented in reports RADC-TR-61-58 and RADC-TDR-62-134, respectively.

We wish to acknowledge the cognizance and direction provided by Mr. Alfred S. Kobos, Advanced Development Laboratory of the Directorate of Communications, Rome Air Development Center, as well as the assistance and encouragement received from other RADC personnel, in particular, Mr. Miles H. Bickelhaupt and Lt. Jack K. Wolf.

Dr. John G. Lawton, Cornell Aeronautical Laboratory, Inc., was project engineer and was assisted by the following CAL personnel, who made major contributions to this effort: Drs. John T. Fleck and William J. Walbesser, and Messrs. Harold D. Becker, Ting T. Chang, Carl F. Evans, Christopher J. Henrich, and Ernest S. Okonski.

Because several relatively independent investigations were performed, each of these is reported as a separate chapter with its own set of equations, figures and references.

PUBLICATION REVIEW

This report has been reviewed and is approved.

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ABSTRACT

This report presents the results of fundamental investigations on a variety of topics related to the optimization of analog and digital data communication systems. The maximum likelihood estimation of FM modulated signals is investigated. A study is made of the threshold phenomenon in FM reception with an ideal discriminator and a postdetection Wiener filter for the case of a random modulation function. Information theory is applied to establish bounds on the performance of analog communications systems. The performance of PCM systems for transmitting analog information is investigated and compared with theoretical bounds for systems of prescribed complexity. Previous work on the partial ordering of digital channels by the criterion of inclusion has been extended. The analysis of the optimization of N-ary digital systems operating over a dispersive channel, which was begun during a previous phase of the contract, is further advanced.

SUMMARY

This report presents the results obtained during the third phase of fundamental investigations in several areas related to the transmission of analog and digital data. While the previous two phases were devoted solely to digital techniques, the present phase is concerned primarily with the optimization of analog demodulation techniques.

The various investigations are reported in six chapters in accordance with the division of the technical effort as follows.

Chapter II reports the results of an investigation of the maximum likelihood estimation of FM modulated signals. The integral equations which describe the maximum likelihood estimation process are developed. The mean square error between the maximum likelihood estimate and the original modulating signal, valid above threshold, are obtained and compared with the mean square error obtained when using a receiver consisting of an ideal descriminator followed by a Wiener postdetection filter.

Chapter III is devoted to an investigation of the threshold phenomenon in the reception of FM signals by a receiver consisting of an ideal discriminator and a postdetection Wiener filter for the case when the modulating function is a random variable.

In Chapter IV the application of information theory to establish bounds on the performance of analog communication systems is discussed.

The performance of PCM systems for transmitting analog information over a digital channel is investigated in Chapter V and compared against bounds on the performance attainable with systems of a prescribed complexity.

Chapter VI extends the work developed in the Phase 2 report on the partial ordering of channels by the criterion of inclusion. This criterion is applied to resolve a paradox observed in the comparison of certain N-ary symmetric channels.

Chapter VII extends the analysis of the optimization of N-ary digital systems operating over a dispersive channel which was begun in the Phase 2 report. Relationships between the transmitted waveforms, the channel transfer function, the spectrum of the noise, and the receiver response function are developed.

Based on the results of this research effort, a number of recommendations for further investigations in areas related to this work are presented.

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I
INTRODUCTION

The present report covers the third phase of investigations of digital and analog communication systems which have been performed by Cornell Aeronautical Laboratory during the period January 1962 through February 1963, under Contract No. AF 30(602)-2210. The objective of this program is to conduct a variety of fundamental investigations for the purpose of assisting the U. S. Air Force in the development of advanced communication systems. The investigations are not aimed at the synthesis or analysis of any particular communication system but, rather, at uncovering the characteristics which govern the behavior of various methods of communication. The entire effort covered by this report was analytic in nature. While the first and second phases of this contract were solely concerned with digital communication systems, the third phase was primarily concerned with analog systems; however, some of our previous work on digital systems has been extended.

This report is organized into seven chapters, each of which treats a particular topic in sufficient completeness so that it may be read independently. Chapters II and III are concerned with reception of randomly modulated FM signals. Chapter II is devoted to maximum likelihood estimation of FM-modulated signals. The maximum likelihood estimate is the a posteriori (i. e., after observation of the received waveform) most likely estimate of the modulating signal. The integral equations which the maximum likelihood estimate must obey are developed. Expressions for the mean square error between the maximum likelihood estimate and the original modulating signal valid above threshold are developed and compared with the mean square error obtained by means of a receiver consisting of an ideal discriminator followed by an optimum (Wiener) linear post-discriminator filter. Chapter III is devoted to an investigation of the threshold phenomenon in the reception of randomly FM-modulated signals by means of the ideal discriminator-Wiener filter receiver.

Chapter IV is devoted to a discussion of the application of information theory to bound the performance of analog communications systems. It is shown that it is not generally possible to specify a maximum attainable output signal-to-noise ratio in terms of the available channel capacity; however, it is possible to bound the maximum attainable ratio of signal entropy power to mean square error between the input and output signals.

In Chapter V the performance of PCM systems for transmitting analog data over a digital channel is investigated. Two forms of binary PCM systems are evaluated in terms of analog signal-to-noise ratios which are obtained as a function of the digital error probabilities. Then the performance of these systems is compared against theoretical bounds on the error rate performance obtainable with digital systems of a specified complexity.

Chapter VI extends the results on the partial ordering of channels by the criterion of inclusion which was developed in the Phase 2 report. This criterion is then applied to resolve a paradox observed in the comparison of certain N-ary symmetric channels.

Chapter VII continues the analysis of optimization of N-ary digital systems operating over a dispersive channel which was originated in the Phase 2 report. While optimization directly in the time domain was attempted in the Phase 2 report, the corresponding conditions which must hold in the frequency domain are developed here. Relationships among the transmitted waveforms, the channel transfer function, the spectrum of the noise, and the receiver response function, which must hold in an optimum system are developed.

II

MAXIMUM LIKELIHOOD RECEPTION OF FM SIGNALS

SUMMARY

In this chapter the application of the method of maximum likelihood to the estimation of intelligence transmitted via frequency modulation is examined. Use of this method for purposes of demodulation was first described by Youla[1] but in the past has been applied only to modulating systems "without memory", that is, to systems such as AM or PM where the present value of the transmitted signal is a function of the present value but not the past of the modulating signal. It is shown that this method can also be applied to modulating systems "with memory" such as FM to yield a pair of nonlinear integral equations, the solution of which specifies the a posteriori most likely estimate $a^*(\tau)$ of the modulating signal $a(\tau)$.

If one assumes that the noise is additive white and gaussian*, the solution of one of the integral equations becomes obvious. The other equation is then simplified by assuming that the carrier frequency is large compared to the bandwidth of the intelligence. It is then further assumed that for sufficiently large signal-to-noise ratios the error, i.e., the difference between the actual intelligence $a(\tau)$ and the maximum likelihood estimate $a^*(\tau)$ goes to zero in a manner which permits linearization of the remaining integral equation.

The linearized integral equation may be solved by the use of a Green's function. The function obtained as the solution of the maximum likelihood problem differs from the modulating signal for two distinct reasons. First, because the "design" is based on the assumption that a certain noise level will be encountered, the output is distorted, even in the absence of any noise. (We are in the position of having taken statistically optimum measures to combat noise and then by chance having received no noise.) Secondly, the output contains a random component due to the random noise actually encountered. The mean square difference between the modulating signal and the demodulated output consists of two statistically independent terms corresponding to these

*The assumption of white gaussian noise is largely motivated by reasons of mathematical expediency.

effects. The mean square error is a function of time τ in the observation interval $t-T \leq \tau \leq t$ as one would expect. Expressions for the mean square error are derived for the zero delay case ($T \rightarrow \infty$, $\tau = t$) and the infinite delay case($\tau - t + T) \rightarrow \infty$, $(t - \tau) \rightarrow \infty$ and compared with similar expressions derived for more conventional FM receivers.

In deriving the integral equations which determine the maximum likelihood estimate, one must, of course, use all available statistical data. It was assumed that these data consist of the autocorrelation functions of the modulating signal and the additive noise and that all other parameters are known. However, it may happen that certain parameters differ from the assumed known values. In order to investigate the effect of such unsuspected parameter variations, the dependence of the mean square error on variation of received signal and noise strength and signal phase has been computed. It is particularly noted that the effects of an initial carrier phase error are attenuated exponentially. For purposes of comparison the mean square error obtained by means of a simplified analytical model, valid at high signal-to-noise ratios, of a conventional FM discriminator followed by an optimum (Wiener) filter are computed. When operating under design conditions the expressions obtained for this case are identical to those obtained for the maximum likelihood reception; however, the sensitivity to deviation from design conditions differ.

INTRODUCTION

The basic reception problem of communication is to obtain at the receiver the "best" estimate of the transmitted intelligence. In order to keep the discussion within the area of communications engineering, we identify intelligence with the modulating waveform produced at the transmitter by the source of intelligence. The data on which this estimate is to be based consists of a finite length record of the received waveform and knowledge of the type of modulation used at the transmitter, the statistics of the intelligence source, and the characteristics of the communications channel. In the simplest case of practical interest the channel is assumed to merely add independent noise to the transmitted waveform.

It is to be noted that the problem as stated above seeks the best estimate of the modulating intelligence directly from the received modulated waveform. Various definitions of best can be employed and will, in general, lead to different estimates. The maximum likelihood solution provides the a posteriori (after utilization of all available data) most likely estimate of the modulating signal. The theory of maximum likelihood reception was first presented by Youla[1]. In that paper it is indicated that the theory can be applied to amplitude and phase modulated systems, and the case of amplitude modulation is treated in some detail. In this paper the application of this theory to frequency modulated systems will be developed. As far as application of Youla's theory is concerned, the most important difference between frequency modulation and amplitude or phase modulation is that with FM the present value of the transmitted waveform depends on the past history as well as upon the present value of the modulating intelligence.

MAXIMUM LIKELIHOOD FM DEMODULATION

1) Derivation of Governing Integral Equations

The system under consideration is shown in Figure 1.

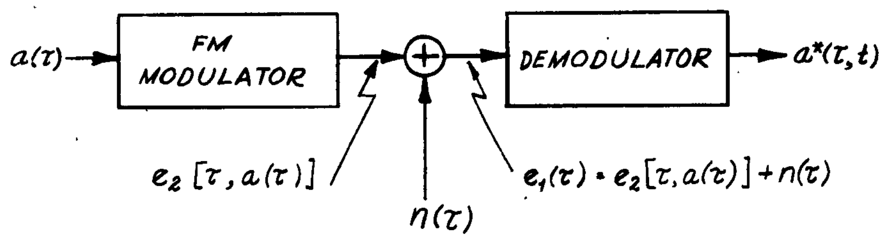


FIGURE 1

Our aim is to develop a demodulator which produces as its output a function which is the most probable estimate of $a(\tau)$, evaluated at time t, given the information from the preceding T seconds where $t - T \leq \tau \leq t$. We define the function $a^*(\tau, t)$ as the most probable $a(\tau)$, given the input $e_1(\tau)$ during the interval $t - T \leq \tau \leq t$ *. For FM we have

$$e_2[\tau, a(\tau)] = E_0 \sin\left[\omega_0 \tau + \beta \int_{t-T}^{\tau} a(u)\,du + \phi\right] ; \quad t - T \leq \tau \leq t \qquad (1)$$

where ϕ represents the unknown carrier phase at $\tau = t - T$, the start of the observation period.

We will assume that both the intelligence $a(\tau)$ and the noise $n(\tau)$ are gaussian processes with zero mean and continuous covariance functions $R_a(s, \tau)$ and $R_n(s, \tau)$.

Then $a(\tau)$ and $n(\tau)$ may be expanded in a Karhunen-Loeve expansion, as follows[2,3]:

*The notation $a^*(\tau,t)$ is used to emphasize the dependence of $a^*(\ )$ on both τ, t. In particular, we shall later compute some properties of $a(t,t)$.

$$a(\tau) = \sum_{i=1}^{\infty} \frac{A_i \phi_i(\tau)}{\lambda_i^{1/2}} \qquad t-T \leq \tau \leq t \tag{2}$$

$$n(\tau) = \sum_{i=1}^{\infty} \frac{N_i \psi_i(\tau)}{\mu_i^{1/2}} \qquad t-T \leq \tau \leq t \tag{3}$$

where the A_i's, N_i are independent gaussianly distributed variables with zero mean and unity variance, and

$$\phi_i(\tau) = \lambda_i \int_{t-T}^{t} R_a(\tau, \xi)\, \phi_i(\xi)\, d\xi \tag{4}$$

$$\psi_i(\tau) = \mu_i \int_{t-T}^{t} R_n(\tau, \xi)\, \psi_i(\xi)\, d\xi \tag{5}$$

The $\phi_i(\tau)$ and the $\psi_i(\tau)$ form two complete orthonormal sets in the interval $t - T \leq \tau \leq t$. Upon the receipt of the waveform $e_1(\tau)$, the ideal receiver can do no more than to compute the a posteriori probability density of all possible intelligence signals $p\left[a(\tau) | e_1(\tau)\right]$.[4] This is, however, not the output one desires from a receiver; what is desired is a single function $a^*(\tau, t)$ which is, in some sense, the best estimate of $a(\tau)$, given the values of $e_1(\tau)$ over the interval $t - T \leq \tau \leq t$. The method of maximum likelihood chooses $a^*(\tau, t)$ such that $p\left[a^*(\tau, t) | e_1(\tau)\right]$ is maximized. This is certainly a reasonable criterion, but one should bear in mind that it is not the only reasonable criterion of optimality. Using Bayes' rule, we have

$$p(a | e_1) = \frac{p(a)\, p(e_1 | a)}{p(e_1)} \tag{6}$$

where $p(a)$ - probability density of $a(\tau)$

$p(a|e_1)$ - conditional density of $a(\tau)$ given $e_1(\tau)$

$p(e_1|a)$ - conditional density of $e_1(\tau)$ given $a(\tau)$

$p(e_1)$ - probability density of $e_1(\tau)$

The integral of a probability density over the entire sample space (e.g., over all possible realizations of the waveform) must be unity. For a particular received waveform $e_1(\tau)$, $p(e_1)$ is a constant, such that $p(a|e_1)$ as given by Equation (6) satisfies this normalization.

We now seek to express Equation (6) in terms of the coordinates $\{A_i\}$ and $\{N_i\}$. The a posteriori most probable signal $a^*(\tau,t)$ is then determined by specification of the a posteriori most probable set $\{A_i^*\}$.

The a priori probability densities of the first K coordinates of $\{A_i\}$ and $\{N_i\}$ are

$$p_K(A_1,\ldots,A_K) = \frac{1}{(2\pi)^{K/2}}\, exp - \frac{1}{2}\sum_{i=1}^{K} A_i^{\,2} \tag{7}$$

$$p_K(N_1,\ldots,N_K) = \frac{1}{(2\pi)^{K/2}}\, exp - \frac{1}{2}\sum_{i=1}^{K} N_i^{\,2} \tag{8}$$

The conditional probability density $p(e_1|a)$ is the probability density of the noise evaluated at $n(\tau) = e_1(\tau) - e_2[\tau, a(\tau)]$. Using Equation (3), the coordinates are

$$N_i = \mu_i^{\,1/2}\int_{t-T}^{t}\left\{e_1(\tau) - e_2\left[\tau, a(\tau)\right]\right\}\psi_i(\tau)\,d\tau = \mu_i^{\,1/2} b_i \tag{9}$$

Substituting Equations (7), (8) and (9) into Equation (6) yields

$$p_K(a|e_1) = C_K\, exp - \frac{1}{2}\sum_{i=1}^{K}(A_i^2 + \mu_i b_i^2) \tag{10}$$

where C_K is a normalizing constant.

The set $\{A^*_{(K)i}\}$ which maximizes $p_K(a|e_1)$ may be found by differentiating Equation (10) with respect to each A_r and requiring that all the derivatives vanish. We shall assume that

$$a_K^*(\tau,t) = \sum_{i=1}^{K} \frac{A^*_{(K)i}\,\phi_i(\tau)}{\lambda_i^{\,1/2}} \qquad t-T \leqslant \tau \leqslant t \tag{11}$$

converges in the mean to the true maximum-likelihood estimate $a^*(\tau,t)$. We find then

$$\frac{\partial p_K(a|e_1)}{\partial A_r} = -C_K\left[A_r+\sum_{i=1}^{K}\mu_i b_i \frac{\partial b_i}{\partial A_r}\right] exp - \frac{1}{2}\sum_{i=1}^{K}(A_i^2+\mu_i b_i^2); \quad (12)$$

for $A_r = A^*_{(K)r}$,

$$0 = A^*_{(K)r} + \sum_{i=1}^{K}\mu_i b_i \frac{\partial b_i}{\partial A^*_{(K)r}} \quad (13)$$

Substituting Equation (1) into Equation (9) and using $a^*_K(\tau,t)$ yields

$$b_i = \int_{t-T}^{t}\left[e_1(x) - E_0 \sin(\omega_0 x + \beta\int_{t-T}^{x} a^*_K(u,t)\,du + \phi)\right]\psi_i(x)\,dx.$$

Using Equation (11) this becomes

$$b_i = \int_{t-T}^{t}\left[e_1(x) - E_0 \sin(\omega_0 x + \beta\int_{t-T}^{x}\sum_{m=1}^{K} A^*_{(K)m}\lambda_m^{-1/2}\phi_m(u)\,du + \phi)\right]\psi_i(x)\,dx \quad (14)$$

$$\frac{\partial b_i}{\partial A^*_{(K)r}} = -E_0\beta\int_{t-T}^{t}\cos\left(\omega_0 x + \beta\int_{t-T}^{x}\sum_{m=1}^{K} A^*_{(K)m}\lambda_m^{-1/2}\phi_m(u)\,du+\phi\right)\int_{t-T}^{x}\lambda_r^{-1/2}\phi_r(u)\,du\,\psi_i(x)\,dx$$

$$= -E_0\beta\int_{t-T}^{t}\cos\left(\omega_0 x + \beta\int_{t-T}^{x} a^*_K(u,t)\,du + \phi\right)\int_{t-T}^{x}\lambda_r^{-1/2}\phi_r(u)\,du\,\psi_i(x)\,dx \quad (15)$$

Substituting Equation (15) into (13)

$$A^*_{(K)r} = E_0\beta\sum_{i=1}^{K}\mu_i b_i\int_{t-T}^{t}\cos(\omega_0 x + \beta\int_{t-T}^{x} a^*_K(u,t)\,du+\phi)\int_{t-T}^{x}\frac{\phi_r(z)}{\lambda_r^{1/2}}\,dz\,\psi_i(x)\,dx \quad (16)$$

Multiplying both sides of Equation (16) by $\lambda_r^{-1/2}\phi_r(\tau)$ and summing with respect to r yields

$$a^*_K(\tau,t) = \sum_{r=1}^{K}\sum_{i=1}^{K}\frac{E_0\beta\phi_r(\tau)\mu_i b_i}{\lambda_r}\int_{t-T}^{t}\cos(\omega_0 x+\beta\int_{t-T}^{x} a^*_K(u,t)\,du+\phi)\int_{t-T}^{x}\phi_r(z)\,dz\,\psi_i(x)\,dx \quad (17)$$

Interchanging both summations with the x integration and the r summation with the z integration yields

$$a_K^*(\tau,t) = E_0\beta\int_{t-T}^{t} \cos\left(\omega_0 x+\beta\int_{t-T}^{x} a_K^*(u,t)\,du+\phi\right)\left(\int_{t-T}^{x}\sum_{r=1}^{K}\frac{\phi_r(z)\phi_r(\tau)}{\lambda_r}dz\right)\left(\sum_{i=1}^{K}\mu_i b_i \psi_i(x)\right)dx \tag{18}$$

We must now consider what happens as $K \longrightarrow \infty$. If $a_K^*(u,t)$ converges in the mean to some function $a^*(u,t)$, then it is easy to see that $\int_{t-T}^{x} a_K^*(u,t)\,du$ converges uniformly in x to $\int_{t-T}^{x} a^*(u,t)\,du$ for $t-T \leq x \leq t$. Therefore, the first expression in the integrand converges uniformly. The second expression converges uniformly to $\int_{t-T}^{x} R_a(z,\tau)dz$ by Mercer's theorem. For the third expression, define

$$g_K(x) = \sum_{i=1}^{K} \mu_i b_i \psi_i(x) \tag{19}$$

Then, it is clear from Equations (5) and (9) that

$$\int_{t-T}^{t} R_n(s,x)\, g_K(x)\,dx = \int_{t-T}^{t}\left\{e_1(x) - e_2\left[x,a_K^*\right]\right\}\sum_{i=1}^{K}\psi_i(x)\,\psi_i(s)\,dx \tag{20}$$

Now it follows from the fact that $\{\psi_i\}$ is an orthonormal complete sequence that the right-hand side of Equation (20) converges in the mean to $e_1(s)-e_2[s,a^*]$ but this does not prove that $\{g_K\}$ converges in the mean. However, it can be shown [5] that, as the eigenvalues λ_i become large, the eigenfunctions ψ_i contain terms of higher and higher frequency. Thus, assuming a bandwidth limited channel, it is reasonable to assume that the higher-order b_i are so small that Equation (19) converges in the mean to a function $g(x)$. We are now justified in letting K go to ∞ in Equations (18) and (20) to get the following integral equations:

$$a^*(\tau,t) = E_0\beta\int_{t-T}^{t}\cos\left(\omega_0 x+\beta\int_{t-T}^{x} a^*(u,t)\,du+\phi\right)\left(\int_{t-T}^{x} R_a(z,\tau)\,dz\right)g(x)\,dx \tag{21}$$

$$e_1(x) - E_0\sin\left(\omega_0 x+\beta\int_{t-T}^{x} a^*(u,t)\,du+\phi\right) = \int_{t-T}^{t} g(z)\,R_n(x,z)\,dz \tag{22}$$

The pair of integral equations (21), (22) specify the operation of a maximum likelihood FM receiver. Note that the maximum likelihood estimate $a^*(\tau,t)$ is determined by all the available data $e_1(s)$, $t-T \leq s \leq t$.

2) Solution of the Integral Equations for the High Signal-to-Noise Ratio Case

If the noise is white $R_n(s,x) = \epsilon_n^2 \delta(s-x)$, Equation (22) has the solution

$$e_1(s) - e_2[s, a^*] = \epsilon_n^2 g(s)$$

where $\epsilon_n^2 = \frac{1}{2}$ noise power density in watts/cps of one-sided spectrum, so that Equation (21) becomes

$$a^*(\tau,t) = \frac{E_0\beta}{\epsilon_n^2}\int_{t-T}^{t} \cos\left(\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right)\left(\int_{t-T}^{x} R_a(z,\tau)\,dz\right)\left[e_1(x) - E_0\sin\left(\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right)\right]dx \quad (23)$$

Equation (23) may be rewritten as

$$a^*(\tau,t) = \frac{E_0\beta}{\epsilon_n^2}\int_{t-T}^{t} \cos\left(\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right) h(x,\tau)\left[E_0\sin\left(\omega_0 x + \beta\int_{t-T}^{x} a(u)\,du + \phi\right) + n(x) - E_0\sin\left(\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right)\right]dx$$

where $h(x,\tau) = \int_{t-T}^{x} R_a(z,\tau)\,dz$

$$\begin{aligned} a^*(\tau,t) = \frac{E_0\beta}{2\epsilon_n^2}\int_{t-T}^{t} h(x,\tau)\Bigg[\Bigg\{ & E_0\sin\left(2\omega_0 x + \beta\int_{t-T}^{x}[a^*(u,t) + a(u)]\,du + 2\phi\right) \\ & - E_0\sin\left(2\omega_0 x + 2\beta\int_{t-T}^{x} a^*(u,t) + 2\phi\right)\Bigg\} + E_0\sin\beta\int_{t-T}^{x}[a(u) - a^*(u,t)]\,du \\ & + 2n(x)\cos\left(\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right)\Bigg]dx \end{aligned} \quad (24)$$

The contribution of the term in $\{\ \}$ may be neglected for ω_0 sufficiently large. Writing $n(x)$ as $n(x) = n_c(x)\cos\omega_0 x + n_s(x)\sin\omega_0 x$, where $n_c(x)$ and $n_s(x)$ are independent white gaussian processes each of intensity $2\epsilon_n^2$ [6] we get

$$\begin{aligned} a^*(\tau,t) = \frac{E_0\beta}{2\epsilon_n^2}\int_{t-T}^{t} h(x,\tau)\Bigg[& E_0\sin\beta\int_{t-T}^{x}[a(u) - a^*(u,t)]\,du + n_c(x)\cos\left(\beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right) \\ & - n_s(x)\sin\left(\beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right) + \Bigg\{ n_c(x)\cos\left(2\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right) \\ & + n_s(x)\sin\left(2\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right)\Bigg\}\Bigg]dx \end{aligned} \quad (25)$$

Again the terms in $\{\ \}$ may be neglected for sufficiently large ω_0 so that

$$a^*(\tau,t) = \frac{E_0\beta}{2\epsilon_n^2}\int_{t-T}^{t} h(x,\tau)\Big[E_0 \sin\beta\int_{t-T}^{x}[a(u)-a^*(u,t)]\,du + n_c(x)\cos\Big(\beta\int_{t-T}^{x}a^*(u,t)\,du+\phi\Big) - n_s(x)\sin\Big(\beta\int_{t-T}^{x}a^*(u,t)\,du+\phi\Big)\Big]dx \tag{26}$$

For sufficiently large signal-to-noise ratios we assume that $a^*(u,t) \to a(u)$ in such a manner that $\left|\beta\int_{t-T}^{x}[a(u)-a^*(u,t)]\,du\right| \ll 1$. This permits Equation (26) to be written in the linearized form:

$$a^*(\tau,t) = \frac{E_0\beta}{2\epsilon_n^2}\int_{t-T}^{t} h(x,\tau)\Big\{E_0\beta\int_{t-T}^{x}[a(u)-a^*(u,t)]\,du + n_c\cos f(x) - n_s \sin f(x)\Big\}\,dx \tag{27}$$

where $f(x) = \beta\int_{t-T}^{x} a(u,t)\,du + \phi$.

Now let us examine the term

$$\eta(x) = n_c \cos f(x) - n_s \sin f(x)$$

Since n_c and n_s are independent gaussian processes, $\eta(x)$ is also a gaussian process. A gaussian process is completely determined by specification of its autocorrelation function

$$\langle\eta(x)\eta(x+\tau)\rangle = \langle\{n_c(x)\cos f(x) - n_s(x)\sin f(x)\}\{n_c(x+\tau)\cos f(x+\tau) - n_s(x+\tau)\sin f(x+\tau)\}\rangle$$

$$\begin{aligned}
&= \langle n_c(x)\, n_c(x+\tau)\rangle \cos f(x)\cos f(x+\tau) \\
&\quad + \langle n_s(x)\, n_s(x+\tau)\rangle \sin f(x)\sin f(x+\tau) \\
&\quad - \langle n_c(x)\, n_s(x+\tau)\rangle \cos f(x)\sin f(x+\tau) \\
&\quad - \langle n_s(x)\, n_c(x+\tau)\rangle \sin f(x)\cos f(x+\tau)
\end{aligned}$$

$$= 2\epsilon_n^2\delta(0)\big[\cos f(x)\cos f(x+\tau) + \sin f(x)\sin f(x+\tau)\big]$$

$$= 2\epsilon_n^2\,\delta(0)$$

so that $\eta(x)$ is just white gaussian noise of intensity $2\epsilon_n^2$.
The integral equation to be solved can now be written

$$a^*(\tau,t) = \frac{E_0^2\beta^2}{2\epsilon_n^2}\int_{t-T}^{t}dx\int_{t-T}^{x}du\int_{t-T}^{x}dv\,R_a(u,\tau)[a(v)-a^*(v,t)] + \frac{E_0\beta}{2\epsilon_n^2}\int_{t-T}^{t}dx\int_{t-T}^{x}du\,R_a(u,\tau)\,\eta(x) \tag{28}$$

Let

$$\zeta(\tau,t) = \frac{E_0\beta}{2\epsilon_n^2}\int_{t-T}^{t} dx \int_{t-T}^{x} du\; R_a(u,\tau)\,\eta(x) \tag{29}$$

$$\begin{aligned} b(u,t) &= a(u,t) - \zeta(u,t) \\ b^*(u,t) &= a^*(u,t) - \zeta(u,t) \end{aligned} \tag{30}$$

then

$$b^*(\tau,t) = \frac{E_0^2\beta^2}{2\epsilon_n^2}\int_{t-T}^{t} dx \int_{t-T}^{x} du \int_{t-T}^{x} dv\; R_a(u,\tau)\left[b(v,t) - b^*(v,t)\right] \tag{31}$$

In order to obtain an explicit solution we assume that

$$R_a(u,\tau) = \frac{1}{2}\, k\epsilon_a^2\, e^{-k|u-\tau|} \tag{32}$$

which corresponds to the power spectrum $S_a(\omega) = \dfrac{\epsilon_a^2 k^2}{\omega^2 + k^2}$.

Note that the mean square value P_a (power) of the intelligence $a(\tau)$ is given by $P_a = R_a(0,0) = \frac{1}{2}k\epsilon_a^2$, we shall eventually normalize both the distortion and the noise in the output to this factor. (The reader who is not particularly interested in the details of the analysis may at this point prefer to go directly to Equation (45).

Substituting Equation (32) and interchanging the order of integration we get

$$\begin{aligned} b^*(\tau,t) = {} & \frac{kE_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{t-T}^{\tau} du \int_{u}^{t} dx \int_{t-T}^{x} dv\; e^{+k(u-\tau)}\left[b(v,t) - b^*(v,t)\right] \\ & + \frac{kE_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{\tau}^{t} du \int_{u}^{t} dx \int_{t-T}^{x} dv\; e^{-k(u-\tau)}\left[b(v,t) - b^*(v,t)\right] \end{aligned} \tag{33}$$

Differentiate twice with respect to τ,

$$\begin{aligned} \frac{\partial b^*(\tau,t)}{\partial\tau} = {} & -\frac{k^2E_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{t-T}^{\tau} du \int_{u}^{t} dx \int_{t-T}^{x} dv\; e^{k(u-\tau)}\left[b(v,t) - b^*(v,t)\right] + \frac{kE_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{\tau}^{t} dx \int_{t-T}^{x} dv\left[b(v,t) - b^*(v,t)\right] \\ & + \frac{k^2E_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{\tau}^{t} du \int_{u}^{t} dx \int_{t-T}^{x} dv\; e^{k(\tau-u)}\left[b(v,t) - b^*(v,t)\right] - \frac{kE_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{\tau}^{t} dx \int_{t-T}^{x} dv\left[b(v,t) - b^*(v,t)\right] \end{aligned} \tag{34}$$

$$\frac{\partial^2 b^*(\tau,t)}{\partial\tau^2} = \frac{E_0^2\beta^2\epsilon_a^2 k^3}{4\epsilon_n^2}\int_{t-T}^{\tau} du\int_u^t dx\int_{t-T}^{x} dv\; e^{k(u-\tau)}\left[b(v,t)-b^*(v,t)\right] - \frac{k^2E_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{\tau}^{t} dx\int_{t-T}^{x} dv\left[b(v,t)-b^*(v,t)\right]$$

$$+\frac{E_0^2\beta^2\epsilon_a^2 k^3}{4\epsilon_n^2}\int_{\tau}^{t} du\int_u^t dx\int_{t-T}^{x} dv\; e^{k(\tau-u)}\left[b(v,t)-b^*(v,t)\right] - \frac{k^2E_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{\tau}^{t} dx\int_{t-T}^{x} dv\left[b(v,t)-b^*(v,t)\right] \tag{35}$$

and we see that

$$\frac{\partial^2 b^*(\tau,t)}{\partial\tau^2} - k^2 b^* = -2\frac{E_0^2\beta^2\epsilon_a^2 k^2}{4\epsilon_n^2}\int_{\tau}^{t} dx\int_{t-T}^{x} dv\left[b(v,t)-b^*(v,t)\right] \tag{36}$$

$$\frac{\partial^3 b^*(\tau,t)}{\partial\tau^3} - k^2\frac{\partial b^*(\tau,t)}{\partial\tau} = 2\frac{E_0^2\beta^2\epsilon_a^2 k^2}{4\epsilon_n^2}\int_{t-T}^{\tau} dv\left[b(v,t)-b^*(v,t)\right] \tag{37}$$

$$\frac{\partial^4 b^*(\tau,t)}{\partial\tau^4} - k^2\frac{\partial^2 b^*(\tau,t)}{\partial\tau^2} = 2\frac{E_0^2\beta^2\epsilon_a^2 k^2}{4\epsilon_n^2}\left[b(\tau,t)-b^*(\tau,t)\right] \tag{38}$$

To get boundary conditions which specify a unique solution of Equation (38), we get from Equations (33) and (34)

$$b^*(t-T,t) = \frac{k E_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{t-T}^{t} du\int_u^t dx\int_{t-T}^{x} dv\; e^{-k(u-t+T)}\left[b(v,t)-b^*(v,t)\right] \tag{39}$$

$$b^*(t,t) = \frac{k E_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{t-T}^{t} du\int_u^t dx\int_{t-T}^{x} dv\; e^{k(u-t)}\left[b(v,t)-b^*(v,t)\right] \tag{40}$$

$$\frac{\partial b^*}{\partial\tau}(t-T,t) = +\frac{k^2E_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{t-T}^{t} du\int_u^t dx\int_{t-T}^{x} dv\; e^{-k(u-t+T)}\left[b(v,t)-b^*(v,t)\right] \tag{41}$$

$$\frac{\partial b^*}{\partial\tau}(t,t) = -\frac{k^2E_0^2\beta^2\epsilon_a^2}{4\epsilon_n^2}\int_{t-T}^{t} du\int_u^t dx\int_{t-T}^{x} dv\; e^{k(u-t)}\left[b(v,t)-b^*(v,t)\right] \tag{42}$$

So $$\frac{\partial b^*}{\partial\tau}(t-T,t) - k\, b^*(t-T,t) = 0; \qquad \frac{\partial b^*}{\partial\tau}(t,t) + k\, b^*(t,t) = 0 \tag{43}$$

Also, from Equations (36) and (37),

$$\frac{\partial^2 b^*}{\partial\tau^2}(t,t) - k^2 b^*(t,t) = 0; \qquad \frac{\partial^3 b^*}{\partial\tau^3}(t-T,t) - k^2\frac{\partial b^*}{\partial\tau}(t-T,t) = 0 \tag{44}$$

The system of Equations (38), (43) and (44) can be solved by means of a Green's function $G(\tau, \xi, t)$, that is, we can set

$$b^*(\tau, t) = \int_{t-T}^{t} G(\tau, \xi, t)\, b(\xi, t)\, d\xi \tag{45}$$

Let us introduce the notation:

$$\lambda = \left(\frac{E_0^2 \beta^2 \epsilon_a^2}{2\epsilon_n^2 k^2}\right)^{1/4} \qquad \text{[dimensionless parameter]} \tag{46}$$

$$L_\tau = \frac{\partial^4}{\partial \tau^4} - k^2 \frac{\partial^2}{\partial \tau^2} + \lambda^4 k^4 \tag{47}$$

$$A_1 f(\tau) = \frac{\partial}{\partial \tau}(t-T) - k f(t-T) \tag{48}$$

$$A_2 f(\tau) = \frac{\partial}{\partial \tau}(t-T) - k^2 \frac{\partial}{\partial \tau}(t-T) \tag{49}$$

$$B_1 f(\tau) = \frac{\partial}{\partial \tau}(t) + k f(t) \tag{50}$$

$$B_2 f(\tau) = \frac{\partial}{\partial \tau}(t) - k^2 f(t) \tag{51}$$

where $f(\tau)$ is an arbitrary, continous function of τ . Then G must satisfy the following:

$$L_\tau G(\tau, \xi; t) = k^4 \lambda^4 \delta(\tau - \xi) \tag{52}$$

$$A_m G(\tau, \xi; t) = 0 = B_m G(\tau, \xi; t) \tag{53}$$

Equation (52) means that $L_\tau G = 0$ where $\tau \neq \xi$, and at $\tau = \xi$, $\partial^3 G(\tau, \xi; t)/\partial \tau^3$ has an upward jump of $k^4 \lambda^4$.

To solve Equations (52) and (53) we shall use matrix notation. First, define C to be the complex number such that

$$C = k\left[\sqrt{\frac{1}{2}\lambda^2 + \frac{1}{4}} + i\sqrt{\frac{1}{2}\lambda^2 - \frac{1}{4}}\right] \tag{54}$$

so that $C^4 - k^2 C^2 + \lambda^4 k^4 = 0$. The elementary solutions of the equation

$L_\tau G = 0$ are arranged in a row matrix:

$$\underset{\sim}{u}(\tau;t) = \left[e^{C(\tau-t)},\ e^{C^*(\tau-t)},\ e^{-C(\tau-t)},\ e^{-C^*(\tau-t)} \right] \tag{55}$$

and we express $G(\tau,\xi;t)$ by means of two column matrices $\underset{\sim}{p}(\xi;t)$ and $\underset{\sim}{q}(\xi;t)$:

$$G(\tau,\xi;t) = \begin{cases} \underset{\sim}{u}(\tau;t)\,\underset{\sim}{p}(\xi;t), & t-T \leq \xi \leq \tau \leq t \\ \underset{\sim}{u}(\tau;t)\,\underset{\sim}{q}(\xi;t), & t-T \leq \tau \leq \xi \leq t \end{cases} \tag{56}$$

Now we define the square matrix

$$\underset{\sim}{W}(\xi;t) = \begin{bmatrix} \underset{\sim}{u}(\xi;t) \\ \underset{\sim}{u}'(\xi;t) \\ \underset{\sim}{u}''(\xi;t) \\ \underset{\sim}{u}'''(\xi;t) \end{bmatrix} \tag{57}$$

where primes denote derivatives with respect to τ ; this is the Wronskian matrix and, hence, is nonsingular. Similarly, define

$$\underset{\sim}{A} = \begin{bmatrix} A_1 \underset{\sim}{u} \\ A_2 \underset{\sim}{u} \\ 0 \\ 0 \end{bmatrix} \qquad \underset{\sim}{B} = \begin{bmatrix} 0 \\ 0 \\ B_1 \underset{\sim}{u} \\ B_2 \underset{\sim}{u} \end{bmatrix} \tag{58}$$

Now Equation (52) is solved for $\tau \neq \xi$ by virtue of Equation (56); for $\tau = \xi$, the continuity conditions can be expressed as

$$\underset{\sim}{W}(\xi;t)\left(\underset{\sim}{p}(\xi;t) - \underset{\sim}{q}(\xi;t)\right) = k^4\lambda^4\,\underset{\sim}{e_4} \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \\ k^4\lambda^4 \end{bmatrix} \tag{59}$$

The boundary conditions can be expressed as

$$\underset{\sim}{A}\,\underset{\sim}{q}(\xi,t) = 0, \qquad \underset{\sim}{B}\,\underset{\sim}{p}(\xi,t) = 0 \tag{60}$$

With a little manipulation we find the solution

$$
\begin{aligned}
\underset{\sim}{p} &= k^4\lambda^4(\underset{\sim}{A}+\underset{\sim}{B})^{-1}\underset{\sim}{A}\,\underset{\sim}{W}^{-1}\underset{\sim}{e}_4 \\
\underset{\sim}{q} &= k^4\lambda^4(\underset{\sim}{A}+\underset{\sim}{B})^{-1}\underset{\sim}{B}\,\underset{\sim}{W}^{-1}\underset{\sim}{e}_4
\end{aligned} \tag{61}
$$

Now

$$
\underset{\sim}{W}(\xi;t) = \begin{bmatrix} e^{c(\xi-t)} & e^{c^*(\xi-t)} & e^{-c(\xi-t)} & e^{-c^*(\xi-t)} \\ ce^{c(\xi-t)} & c^*e^{c^*(\xi-t)} & -ce^{-c(\xi-t)} & -c^*e^{-c^*(\xi-t)} \\ c^2e^{c(\xi-t)} & c^{*2}e^{c^*(\xi-t)} & c^2e^{-c(\xi-t)} & c^{*2}e^{-c^*(\xi-t)} \\ c^3e^{c(\xi-t)} & c^{*3}e^{c^*(\xi-t)} & -c^3e^{-c(\xi-t)} & -c^{*3}e^{-c^*(\xi-t)} \end{bmatrix} \tag{62}
$$

$$
\underset{\sim}{A} = \begin{bmatrix} (c-k)e^{-cT} & (c^*-k)e^{-c^*T} & -(c+k)^{cT} & -(c^*+k)e^{c^*T} \\ (c^3-k^2c)e^{-cT} & (c^{*3}-k^2c)^{-c^*T} & -(c^3-k^2c)e^{cT} & -(c^{*3}-k^2c^*)e^{c^*T} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{63}
$$

$$
\underset{\sim}{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (c+k) & c^*+k & -(c-k) & -(c^*-k) \\ c^2-k^2 & c^{*2}-k^2 & c^2-k^2 & c^{*2}-k^2 \end{bmatrix} \tag{64}
$$

Now if the real part of cT , that is, $kT\sqrt{\frac{1}{2}\lambda^2+1/4}$, is sufficiently large, the entries in the first two columns of $\underset{\sim}{A}$ approach zero. Thus, using $\underset{\sim}{A}_{(2)}$, $\underset{\sim}{B}_{(1)}$, and $\underset{\sim}{B}_{(2)}$ to denote appropriate submatrices,

$$
\underset{\sim}{A} = \left[\begin{array}{c|c} 0 & \underset{\sim}{A}_{(2)} \\ \hline 0 & 0 \end{array}\right], \quad \underset{\sim}{B} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \underset{\sim}{B}_{(1)} & \underset{\sim}{B}_{(2)} \end{array}\right], \quad \underset{\sim}{A}+\underset{\sim}{B} = \left[\begin{array}{c|c} 0 & \underset{\sim}{A}_{(2)} \\ \hline \underset{\sim}{B}_{(1)} & \underset{\sim}{B}_{(2)} \end{array}\right] \tag{65}
$$

$$
(\underset{\sim}{A}+\underset{\sim}{B})^{-1} = \left[\begin{array}{c|c} -\underset{\sim}{B}_{(1)}^{-1}\underset{\sim}{B}_{(2)}\underset{\sim}{A}_{(2)}^{-1} & \underset{\sim}{B}_{(1)}^{-1} \\ \hline \underset{\sim}{A}_{(2)}^{-1} & 0 \end{array}\right], (\underset{\sim}{A}+\underset{\sim}{B})^{-1}\underset{\sim}{A} = \begin{bmatrix} 0 & -\underset{\sim}{B}_{(1)}^{-1}\underset{\sim}{B}_{(2)} \\ 0 & 1 \end{bmatrix}, (\underset{\sim}{A}+\underset{\sim}{B})^{-1}\underset{\sim}{B} = \begin{bmatrix} 1 & \underset{\sim}{B}_{(1)}^{-1}\underset{\sim}{B}_{(2)} \\ 0 & 0 \end{bmatrix} \tag{66}
$$

A more detailed analysis shows that Equation (66) is correct to within an error of order $|e^{-2cT}|$. In effect, then, we have let T tend to ∞ .

We now find

$$\underset{\sim}{W}^{-1} = \frac{1}{2CC^*(C^2-C^{*2})}\begin{bmatrix} -CC^{*3}e^{-C(\xi-t)} & -C^*e^{-C(\xi-t)} & +CC^*e^{-C(\xi-t)} & +C^*e^{-C(\xi-t)} \\ +C^3C^*e^{-C^*(\xi-t)} & +C^3e^{-C^*(\xi-t)} & -CC^*e^{-C^*(\xi-t)} & -Ce^{-C^*(\xi-t)} \\ -CC^{*3}e^{+C(\xi-t)} & +C^{*3}e^{C(\xi-t)} & +CC^*e^{+C(\xi-t)} & -C^*e^{C(\xi-t)} \\ +C^3C^*e^{+C^*(\xi-t)} & -C^3e^{C^*(\xi-t)} & -CC^*e^{+C^*(\xi-t)} & +Ce^{C^*(\xi-t)} \end{bmatrix} \tag{67}$$

$$\underset{\sim}{B}_{(1)} = \begin{bmatrix} 1 & 1 \\ C-k & C^*-k \end{bmatrix}\begin{bmatrix} C+k & 0 \\ 0 & C^*+k \end{bmatrix} \qquad \underset{\sim}{B}_{(1)}^{-1} = \frac{1}{(C-C^*)(C+k)(C^*+k)}\begin{bmatrix} C^*+k & 0 \\ 0 & C+k \end{bmatrix}\begin{bmatrix} -(C^*-k) & 1 \\ C-k & -1 \end{bmatrix} \tag{68}$$

$$\underset{\sim}{B}_{(1)}^{-1}\underset{\sim}{B}_{(2)} = \frac{1}{(C-C^*)(C+k)(C^*+k)}\begin{bmatrix} (C+C^*)(C^*+k)(C-k) & 2C^*(C^*+k)(C^*-k) \\ -2C(C+k)(C-k) & -(C+C^*)(C+k)(C^*-k) \end{bmatrix} \tag{69}$$

$$k^4\lambda^4\underset{\sim}{W}^{-1}\underset{\sim}{e}_4 = C^2C^{*2}\underset{\sim}{W}^{-1}\underset{\sim}{e}_4 = \frac{CC^*}{2(C^2-C^{*2})}\begin{bmatrix} C^*e^{-C(\xi-t)} \\ -Ce^{-C^*(\xi-t)} \\ -C^*e^{C(\xi-t)} \\ Ce^{C^*(\xi-t)} \end{bmatrix} \tag{70}$$

$$(\underset{\sim}{A}+\underset{\sim}{B})^{-1}\underset{\sim}{A} = \frac{1}{(C-C^*)(C+k)(C^*+k)}\begin{bmatrix} 0 & 0 & -(C+C^*)(C^*+k)(C-k) & -2C^*(C^*+k)(C^*-k) \\ 0 & 0 & 2C(C+k)(C-k) & (C+C^*)(C+k)(C^*-k) \\ 0 & 0 & (C-C^*)(C+k)(C^*+k) & 0 \\ 0 & 0 & 0 & (C-C^*)(C+k)(C^*+k) \end{bmatrix} \tag{71}$$

$$(\underset{\sim}{A}+\underset{\sim}{B})^{-1}\underset{\sim}{B} = \frac{1}{(C-C^*)(C+k)(C^*+k)}\begin{bmatrix} -(C-C^*)(C+k)(C^*+k) & 0 & -(C+C^*)(C^*+k)(C-k) & -2C^*(C^*+k)(C^*-k) \\ 0 & -(C-C^*)(C+k)(C^*-k) & 2C(C+k)(C-k) & (C+C^*)(C+k)(C^*-k) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{72}$$

Solving for $G(\tau, \xi; t)$ one obtains

$$G(\tau,\xi;t)=\frac{CC^*}{2(C^2-C^{*2})(C-C^*)(C+k)(C^*+k)}\Big[+C^*(C+C^*)(C^*+k)(C-k)e^{-C(2t-\xi-\tau)}$$
$$+C(C+C^*)(C+k)(C^*-k)e^{-C^*(2t-\xi-\tau)}-2CC^*(C^*+k)(C^*-k)e^{-(C+C^*)t+C^*\xi+C\tau}$$
$$-2CC^*(C+k)(C-k)e^{-(C+C^*)t+C\xi+C^*\tau}-C^*(C-C^*)(C+k)(C^*+k)e^{-C|\tau-\xi|}$$
$$+C(C-C^*)(C+k)(C^*+k)e^{-C^*|\tau-\xi|}\Big] \tag{73}$$

To recapitulate,

$$b^*(\tau;t)=\int_{-\infty}^{t}G(\tau,\xi;t)\,b(\xi;t)\,d\xi \tag{45}$$

or

$$a^*(\tau;t)=\zeta(\tau;t)+\int_{-\infty}^{t}G(\tau,\xi;t)\Big[a(\xi)-\zeta(\xi;t)\Big]d\xi \tag{74}$$

with ζ given by Equation (29) and G by (73).

Now $t-\tau$ may be considered as the "delay time" of the demodulator which these equations describe. In particular, we shall consider the cases of zero delay and infinite delay:

Zero delay: $t=\tau$: $G(\tau,\xi;\tau)=G_0(\tau-\xi)=$

$$\frac{C^2C^{*2}}{(C-C^*)(C+k)(C^*+k)}\Big[-e^{-C(\tau-\xi)}+e^{-C^*(\tau-\xi)}\Big] \tag{75}$$

Infinite delay: $t\to\infty$: $G(\tau,\xi;\infty)=G_\infty(\tau-\xi)=$

$$\frac{CC^*}{2(C^2-C^{*2})}\Big[-C^*e^{-C|\tau-\xi|}+Ce^{-C^*|\tau-\xi|}\Big] \tag{76}$$

It will prove useful to note from Equation (54) that

$$C^2+C^{*2}=k^2 \tag{77a}$$

Alternatively,

$$(C+k)(C-k) = -C^{*2} \tag{77b}$$

$$(C^*+k)(C^*-k) = -C^2 \tag{77c}$$

also

$$CC^* = k^2\lambda^2 \tag{77d}$$

so that

$$G_0(\tau-\xi) = \frac{(C-k)(C^*-k)}{(C-C^*)}\left[-e^{-C(\tau-\xi)}+e^{-C^*(\tau-\xi)}\right] \tag{78}$$

$$G_\infty(\tau-\xi) = \frac{1}{2(C^2-C^{*2})}\left[C(C^2-k^2)e^{-C|\tau-\xi|}-C^*(C^{*2}-k^2)e^{-C^*|\tau-\xi|}\right] \tag{79}$$

3) Derivation of the Mean Square Error and Its Sensitivity to Off-Design Conditions

It is of interest to evaluate the mean square difference between the function $a^*(\tau,t)$ obtained as the solution of Equation (23) by the methods discussed above, and the modulating function $a(\tau)$. We note that Equation (23) was derived on the assumption that the difference between the received and transmitted signal consists only of additive noise of known intensity. It seems unrealistic to assume that the strength and the phase of the received carrier or the intensity of the additive noise are known precisely. We shall, therefore, determine the dependence of the mean square error on variations of these parameters. Two observations can be made at this point: (1) the function $a^*(\tau,t)$ was obtained as the maximum likelihood solution on the assumption that these parameters are known exactly, (2) the maximum likelihood solution is not, in general, the least mean square solution[6].

We will now investigate the mean square error of a detection system which operates in accordance with Equation (23) derived on the assumption that E_o, ϕ and ϵ_n are known exactly but instead receives $\hat{E}_o, \hat{\phi} = \phi + \Delta\phi$ and $\hat{\epsilon}_n$. The determination of the sensitivity to deviations from design parameters is particularly important in cases such as this where the system has been optimized in a fairly abstract manner, for example, variations of the phase can be disastrous to a coherent PSK system.

Let

$$e_1(\tau) = \hat{E}_o \sin\left(\omega_o \tau + \beta \int_{t-T}^{\tau} a(u)\,du + \phi + \Delta\phi\right) + \hat{n}(\tau) \tag{80}$$

where $\hat{E}_o = \mu_s E_o$ · · · · · · · · · · strength of received carrier

$\hat{n}(\tau) = \mu_n n(\tau)$ · · · · · · · · · · receiver noise of intensity $\hat{\epsilon}_n^2 = \mu_n^2 \epsilon_n^2$

The three parameters, μ_s , μ_n , $\Delta\phi$ are introduced here to account for "off-design" operation.

Substituting Equation (80) into (23)

$$a^*(\tau,t) = \frac{E_0\beta}{\epsilon_n^2}\int_{t-T}^{t}\cos\left(\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right)\left(\int_{t-T}^{x} R_a(z,\tau)\,dz\right) \tag{81}$$
$$\left[\mu_s E_0 \sin\left(\omega_0 x + \beta\int_{t-T}^{x} a(u)\,du + \phi + \Delta\phi\right) + \mu_n n(x) - E_0 \sin\left(\omega_0 x + \beta\int_{t-T}^{x} a^*(u,t)\,du + \phi\right)\right]dx$$

We will first assume that $\Delta\phi = 0$ and later consider the case $\Delta\phi \neq 0$ separately. By repeating the arguments which led from Equation (23) to Equation (28) one obtains

$$a^*(\tau,t) = \frac{\mu_s E_0^2\beta^2}{2\epsilon_n^2}\int_{t-T}^{t} dx\int_{t-T}^{x} dz\int_{t-T}^{x} du\, R_a(z,\tau)\left[a(u) - a^*(u)\right] \tag{82}$$
$$+ \frac{\mu_n E_0\beta}{2\epsilon_n^2}\int_{t-T}^{t} dx\int_{t-T}^{x} dz\, R_a(z,\tau)\,\eta(x)$$

where $\eta(x)$ has the same statistics as before: white gaussian noise with autocorrelation function $2\epsilon_n^2\delta(\tau)$. The solution of Equation (82) is given by Equations (74) and (73), with these changes: λ is redefined as

$$\lambda = \left[\frac{\mu_s E_0^2\beta^2\epsilon_a^2}{2k^2\epsilon_n^2}\right]^{1/4} \tag{83}$$

and

$$\zeta = \frac{\mu_n E_0\beta}{2\epsilon_n^2}\int_{t-T}^{t} dx\int_{t-T}^{x} dz\, R_a(z,\tau)\,\eta(x) \tag{84}$$

From Equation (74) we see that the solution of Equation (82) can be symbolized:

$$a^* = \zeta + g(a-\zeta) = ga + (1-g)\zeta$$

The difference $a - a^*$ can then be written as

$$a - a^* = (1-g)\,a - (1-g)\,\zeta \tag{85}$$

While both terms are stochastic, they are independent since the first, or distortion term, is a function of the signal $a(\tau)$ only while the second, or noise term, is a function of the noise $n(\tau)$ only. The mean square error is, therefore, given by the sums of the mean squares of the separate parts which we will now compute for the case of zero and infinite delay.

a) Zero Delay

(1) Distortion term, D_0.

$$\begin{aligned} g_a(t,t) &= \int_{-\infty}^{t} G(t,\xi,t)\, a(\xi)\, d\xi \\ &= \int_0^{\infty} G_0(\xi)\, a(t-\xi)\, d\xi \end{aligned} \tag{86}$$

$a(t)$ may be obtained by passing white gaussian noise $\alpha(t)$ having intensity ϵ_a^2 through a filter with transfer function $|F(\omega)|^2 = \frac{k^2}{k^2+\omega^2}$ so that

$$a(t) = k\int_0^{\infty} e^{-k\tau} \alpha(t-\tau)\, d\tau \tag{87}$$

$$\begin{aligned} g_a(t,t) - a(t) &= \int_0^{\infty} d\xi \int_0^{\infty} d\tau\, k G_0(\xi)\, e^{-k\tau} \alpha(t-\xi-\tau) - \int_0^{\infty} d\tau\, k e^{-k\tau} \alpha(t-\tau) \\ &= \int_0^{\infty} d\tau \left[\int_0^{\tau} d\xi\, G_0(\xi)\, k e^{-k(\tau-\xi)} - k e^{-k\tau} \right] \alpha(t-\tau). \end{aligned} \tag{88}$$

$$k\int_0^{\tau} d\xi\, G_0(\xi)\, e^{-k(\tau-\xi)} = \int_0^{\tau} \frac{k(C-k)(C^*-k)}{C-C^*} \left[-e^{-(C-k)\xi - k\tau} + e^{-(C^*-k)\xi - k\tau} \right] d\xi$$

$$= \frac{k}{C-C^*} \left[-(C^*-k)e^{-k\tau} + (C^*-k)e^{-C\tau} + (C-k)e^{-k\tau} - (C-k)e^{-C^*\tau} \right]$$

$$= k e^{-k\tau} + \frac{k(C^*-k)}{(C-C^*)} e^{-C\tau} - \frac{k(C-k)}{(C-C^*)} e^{-C^*\tau} \tag{89}$$

$$g_a(t,t) - a(t) = \int_0^{\infty} d\tau \left[\frac{k(C^*-k)}{C-C^*} e^{-C\tau} - \frac{k(C-k)}{C-C^*} e^{-C^*\tau} \right] \alpha(t-\tau) \tag{90}$$

The mean square of the distortion term,

$$D_0 \equiv \langle (\mathcal{G}a(t,t) - a(t))^2\rangle = \epsilon_a^2 \int_0^\infty d\tau \left[\frac{k(C^*-k)}{C-C^*} e^{-C\tau} - \frac{k(C-k)}{C-C^*} e^{-C^*\tau}\right]^2$$

$$= \epsilon_a^2 \int_0^\infty d\tau \left[\frac{k^2(C^*-k)^2}{(C-C^*)^2} e^{-2C\tau} - \frac{2k^2(C-k)(C^*-k)}{(C-C^*)^2} e^{-(C+C^*)\tau} + \frac{k^2(C-k)^2}{(C-C^*)^2} e^{-2C^*\tau}\right]$$

$$= \epsilon_a^2 \cdot \frac{k^2}{(C-C^*)^2}\left[\frac{(C^*-k)^2}{2C} - \frac{2(C-k)(C^*-k)}{C+C^*} + \frac{(C-k)^2}{2C^*}\right]$$

$$= \frac{k^2\epsilon_a^2}{(C-C^*)^2}\left[(C^*-k)\left(\frac{C^*-k}{2C} - \frac{C-k}{C+C^*}\right) - (C-k)\left(\frac{C^*-k}{C+C^*} - \frac{C-k}{2C^*}\right)\right]$$

$$= \frac{k^2\epsilon_a^2}{(C-C^*)^2}\left[(C^*-k)\frac{(C+C^*)(C^*-k)-2C(C-k)}{2C(C+C^*)} - (C-k)\frac{(C^*-k)2C^*-(C+C^*)(C-k)}{2C^*(C+C^*)}\right]$$

$$= \frac{k^2\epsilon_a^2}{(C-C^*)^2}\left[(C^*-k)\frac{-2C^2+CC^*+C^{*2}+kC-kC^*}{2C(C+C^*)} + (C-k)\frac{C^2+CC^*-2C^{*2}-kC+kC^*}{2C^*(C+C^*)}\right]$$

$$= \frac{k^2\epsilon_a^2}{(C-C^*)}\left[\frac{(C^*-k)(-2C-C^*+k)}{2C(C+C^*)} + \frac{(C-k)(C+2C^*-k)}{2C^*(C+C^*)}\right]$$

$$= \frac{k^2\epsilon_a^2}{(C-C^*)}\left[\frac{-2(C+C^*)(C^*-k)+(C^*+k)(C^*-k)}{2C(C+C^*)} + \frac{2(C+C^*)(C-k)-(C-k)(C+k)}{2C^*(C+C^*)}\right]$$

$$= \frac{k^2\epsilon_a^2}{(C-C^*)}\left[-\frac{C^*-k}{C} - \frac{C^2}{2C(C+C^*)} + \frac{(C-k)}{C^*} + \frac{C^{*2}}{2C^*(C+C^*)}\right]$$

$$= \frac{k^2\epsilon_a^2}{C-C^*}\left[\frac{-C^*(C^*-k)+C(C-k)}{CC^*} - \frac{C-C^*}{2(C+C^*)}\right] = k^2\epsilon_a^2\left[\frac{C+C^*-k}{CC^*} - \frac{1}{2(C+C^*)}\right]$$

$$= k^2\epsilon_a^2 \cdot \frac{2(C+C^*)^2 - 2k(C+C^*) - CC^*}{2CC^*(C+C^*)} = \frac{k^2\epsilon_a^2\left[3CC^* - 2k(C+C^*) + 2k^2\right]}{2CC^*(C+C^*)}.$$

From Equation (54)

$$CC^* = k^2\lambda^2, \quad C+C^* = k\sqrt{2\lambda^2+1},$$

so that

$$D_0 = \frac{1}{2} k\epsilon_a^2 \cdot \frac{3\lambda^2+2-2\sqrt{2\lambda^2+1}}{\lambda^2\sqrt{2\lambda^2+1}}$$

$$= \frac{3\lambda^2+2-2\sqrt{2\lambda^2+1}}{\lambda^2\sqrt{2\lambda^2+1}} P_a \sim \frac{3}{\sqrt{2}}\lambda^{-1} P_a, \text{ for large } \lambda. \qquad (91)$$

The mean square distortion, D_0, is determined by the product of the intelligence power $P_a = \frac{1}{2} k \epsilon_a^2$ and the function $f(\lambda) = \frac{3\lambda^2 + 2 - 2\sqrt{2\lambda^2+1}}{\lambda^2\sqrt{2\lambda^2+1}}$. It will be noted that $f(\lambda)$ decreases monotonically to zero with increasing λ.

The value of the dimensionless parameter $\lambda = \left(\frac{\mu_3 E_0^2 \beta^2 \epsilon_a^2}{2k^2\epsilon_n^2}\right)^{1/4}$ provides a quantitative measure of the channel quality. It seems worthwhile to explore the relationship of λ to the more conventional FM parameters of carrier-to-noise power ratio and modulation index. In the usual FM analysis sinusoidal modulation at frequency f_a is assumed. The discriminator output is passed through an ideal, rectangular, low-pass filter and the noise power is computed in that bandwidth. The modulation index, M, is defined as the ratio of carrier frequency deviation, Δf, to modulating frequency f_a. In order to apply these concepts to the situation at hand, we define a noise equivalent bandwidth B_N and the modulation index M as follows: Let B_N be the bandwidth of an ideal, rectangular, low-pass filter, having the same area as the intelligence power spectrum*.

$$B_N = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{k^2}{k^2+\omega^2}\, d\omega = \frac{k}{2}\ cps \qquad \text{(double-sided spectrum)}$$

The noise power N in the bandwidth B_N is then

$$N = \epsilon_n^2 B_N = \frac{1}{2} k \epsilon_n^2 \quad \cdots\cdots\cdots \text{ RF noise in intelligence bandwidth } B_N .$$

The modulation index M is defined by

$$M^2 = \frac{\text{Mean Square frequency deviation due to modulation}}{\text{Square of noise equivalent bandwidth}} = \frac{\langle(\Delta f)^2\rangle}{B_N^2}$$

$$= \left(\frac{\beta}{2\pi B_N}\right)^2 \langle a^2 \rangle = \frac{\beta^2 \epsilon_a^2}{(2\pi)^2 B_N} .$$

*The intelligence signal $a(\tau)$ can be obtained by passing white gaussian noise, of intensity ϵ_n^2, through a filter with transfer function $|F(\omega)|^2 = k^2/(\omega^2 + k^2)$, the intelligence power is then $P_a = \frac{1}{2}(k\epsilon_a^2)$. An ideal filter of bandwidth B_N will also have output power P_a; hence, B_N is called the noise equivalent bandwidth.

Then $\lambda^4 = \dfrac{\mu_s E_0^2 \beta^2 \epsilon_a^2}{2 k^2 \epsilon_n^2} = \mu_s (\pi M)^2 \dfrac{P_c}{N}$ (92)

where $P_c = \dfrac{1}{2} E_0^{\,2} \ldots$ Design Carrier Power

(2) Noise term, N_0.

In order to obtain the steady-state noise term of the zero delay estimate we let $T \to \infty$ in Equation (84).

$$\begin{aligned}
\zeta(\tau, t) &= \frac{\mu_n E_0 \beta}{2\epsilon_n^2} \int_{-\infty}^{t} dx \int_{-\infty}^{x} dz\, R_a(z, \tau)\, \eta(x) \\
&= \frac{\mu_n E_0 \beta k \epsilon_a^2}{4\epsilon_n^2} \int_{-\infty}^{t} dx \int_{-\infty}^{x} dz\, e^{-k|z-\tau|}\, \eta(x) \qquad (93) \\
&= \frac{\mu_n \epsilon_a k^2 \lambda^2}{2^{3/2} \mu_s^{1/2} \epsilon_n} \int_{-\infty}^{t} dx \int_{-\infty}^{x} dz\, e^{-k|z-\tau|}\, \eta(x)
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}\zeta(t, t) &= \int_{0}^{\infty} d\xi\, G_0(\xi)\, \zeta(t-\xi, t) \\
&= \frac{\mu_n \epsilon_a k^2 \lambda^2}{\epsilon_n \sqrt{8\mu_s}} \int_{0}^{\infty} d\xi \int_{-\infty}^{t} dx \int_{-\infty}^{x} dz\, G_0(\xi)\, e^{-k|z-t+\xi|}\, \eta(x) \qquad (94) \\
&= \frac{\mu_n \epsilon_a k^2 \lambda^2}{\epsilon_n \sqrt{8\mu_s}} \int_{-\infty}^{t} dx \left[\int_{0}^{\infty} d\xi \int_{-\infty}^{x} dz\, G_0(\xi)\, e^{-k|z-t+\xi|} \right] \eta(x)
\end{aligned}$$

$$\mathcal{G}\zeta(t, t) - \zeta(t, t) = \frac{\mu_n \epsilon_a k^2 \lambda^2}{\epsilon_n \sqrt{8\mu_s}} \int_{-\infty}^{t} dx \left[\int_{0}^{\infty} d\xi \int_{-\infty}^{x} dz\, G_0(\xi) e^{-k|z-t+\xi|} - \int_{-\infty}^{x} dz\, e^{-k|z-t|} \right] \eta(x) \qquad (95)$$

$$\begin{aligned}
&\int_{0}^{\infty} d\xi \int_{-\infty}^{x} dz\, G_0(\xi)\, e^{-k|z-t+\xi|} - \int_{-\infty}^{x} dz\, e^{-k|z-t|} \\
&= \int_{-\infty}^{x} dz \left[\int_{0}^{\infty} d\xi\, G_0(\xi) e^{-k|z-t+\xi|} - e^{k(z-t)} \right] \qquad (96)
\end{aligned}$$

Note $z \le x \le t$

Since $\eta(x)$ is stationary, it is clear that the statistics of the noise term are independent of t, and we will therefore let $t = 0$.

$$\int_0^\infty d\xi G_0(\xi) e^{-k|z+\xi|} = \int_0^{-z} d\xi G_0(\xi) e^{k(z+\xi)} + \int_{-z}^\infty d\xi G_0(\xi) e^{-k(z+\xi)} \qquad (z<0)$$

$$= \frac{(C-k)(C^*-k)}{C-C^*} \int_0^{-z} d\xi \left[-e^{-(C-k)\xi+kz} + e^{-(C^*-k)\xi+kz} \right]$$

$$+ \frac{(C-k)(C^*-k)}{C-C^*} \int_{-z}^\infty d\xi \left[-e^{-(C+k)\xi-kz} + e^{-(C^*+k)\xi-kz} \right]$$

$$= \frac{1}{C-C^*} \left[-(C^*-k)e^{kz} + (C^*-k)e^{Cz} + (C-k)e^{kz} - (C-k)e^{C^*z} \right] \tag{97}$$

$$+ \frac{(C-k)(C^*-k)}{C-C^*} \left[-(C+k)^{-1} e^{Cz} + (C^*+k)^{-1} e^{C^*z} \right]$$

$$= e^{kz} + \frac{(C^*-k)}{C-C^*}\left(1 - \frac{C-k}{C+k}\right) e^{Cz} - \frac{C-k}{C-C^*}\left[1 - \frac{C^*-k}{C^*+k}\right] e^{C^*z}$$

$$= e^{kz} + \frac{2k(C^*-k)}{(C+k)(C-C^*)} e^{Cz} - \frac{2k(C-k)}{(C-C^*)(C^*+k)} e^{C^*z}$$

$$\int_{-\infty}^x dz \left[\int_0^\infty d\xi G_0(\xi) e^{-k|z+\xi|} - e^{kz} \right]$$

$$= \int_{-\infty}^x dz \left[\frac{2k(C^*-k)}{(C+k)(C-C^*)} e^{Cz} - \frac{2k(C-k)}{(C-C^*)(C^*+k)} e^{C^*z} \right] \tag{98}$$

$$= \frac{2k(C^*-k)}{C(C+k)(C-C^*)} e^{Cx} - \frac{2k(C-k)}{C^*(C^*+k)(C-C^*)} e^{C^*x}$$

Substituting Equation (98) into (95) and recalling $\langle \eta(x)\eta(x+\tau)\rangle = 2\epsilon_n^2 \delta(\tau)$, one obtains

$$N_0 \equiv \langle (\mathcal{G}\zeta(t,t) - \zeta(t,t))^2 \rangle$$

$$= \frac{\mu_n^2 \epsilon_a^2 k^4 \lambda^4}{4\mu_s} \int_{-\infty}^0 dx \left[\frac{2k(C^*-k)e^{Cx}}{C(C+k)(C-C^*)} - \frac{2k(C-k)e^{C^*x}}{C^*(C^*+k)(C-C^*)} \right]^2 \tag{99}$$

$$N_o = \frac{\mu_n^2 \epsilon_a^2 k^4 \lambda^4}{4\mu_s} \int_{-\infty}^{0} dx \left[\frac{4k^2(C^*-k)^2 e^{2Cx}}{C^2(C+k)^2(C-C^*)^2} - \frac{8k^2(C-k)(C^*-k)e^{(C+C^*)x}}{CC^*(C+k)(C^*+k)(C-C^*)^2} + \frac{4k^2(C-k)^2 e^{2C^*x}}{C^{*2}(C^*+k)^2(C-C^*)^2}\right]$$

$$= \frac{\mu_n^2 \epsilon_a^2 C^2 C^{*2}}{4\mu_s}\left[\frac{2k^2(C^*-k)^2}{C^3(C+k)^2(C-C^*)^2} - \frac{8k^2(C-k)(C^*-k)}{CC^*(C+C^*)(C-C^*)^2(C+k)(C^*+k)} + \frac{2k^2(C-k)^2}{C^{*3}(C^*+k)^2(C-C^*)^2}\right]$$

$$= \frac{\mu_n^2 \epsilon_a^2}{4\mu_s}\left[\frac{2k^2(C-k)^2(C^*-k)^2}{CC^{*2}(C-C^*)^2} - \frac{8k^2(C-k)^2(C^*-k)^2}{CC^*(C+C^*)(C-C^*)^2} + \frac{2k^2(C-k)^2(C^*-k)^2}{C^2C^*(C-C^*)^2}\right]$$

$$= \frac{\mu_n^2 \epsilon_a^2 k^2 (C-k)^2(C^*-k)^2}{2\mu_s C^2 C^{*2}(C+C^*)(C-C^*)^2}\left[C(C+C^*) - 4CC^* + C^*(C+C^*)\right]$$

$$= \frac{\mu_n^2 \epsilon_a^2 k^2 (C-k)^2(C^*-k)^2}{2\mu_s C^2 C^{*2}(C+C^*)} = \frac{\mu_n^2 \epsilon_a^2 k^2}{2\mu_s C^2 C^{*2}(C+C^*)}\left[CC^* - k(C+C^*) + k^2\right]^2$$

$$= \frac{\mu_n^2 \epsilon_a^2 k^2}{2\mu_s C^2 C^{*2}(C+C^*)}\left[C^2C^{*2} - 2kCC^*(C+C^*) + 2k^2CC^* + k^2(C+C^*)^2 - 2k^3(C+C^*) + k^4\right]$$

$$= \frac{\mu_n^2 \epsilon_a^2 k^2}{2\mu_s C^2 C^{*2}(C+C^*)}\left[C^2C^{*2} - 2kCC^*(C+C^*) + 4k^2CC^* - 2k^3(C+C^*) + 2k^4\right]$$

$$= \frac{1}{2} k\epsilon_a^2 \cdot \frac{\mu_n^2}{\mu_s} \cdot \frac{\lambda^4 + 4\lambda^2 + 2 - (2\lambda^2+2)\sqrt{2\lambda^2+1}}{\lambda^4\sqrt{2\lambda^2+1}} \tag{100}$$

$$= \frac{\mu_n^2}{\mu_s} P_a \frac{\lambda^4 + 4\lambda^2 + 2 - (2\lambda^2+2)\sqrt{2\lambda^2+1}}{\lambda^4\sqrt{2\lambda^2+1}}$$

For large λ, this is approximately

$$N_o = \frac{\mu_n^2}{\mu_s} P_a \frac{1}{\sqrt{2}\,\lambda}. \tag{101}$$

The mean square noise term, N_o, is obtained as the product of the intelligence power $P_a = k\epsilon_a^2/2$ and the term $\frac{\mu_n^2}{\mu_s} g(\lambda)$ where

$$g(\lambda) = \frac{\lambda^4 + 4\lambda^2 + 2 - (2\lambda^2+2)\sqrt{2\lambda^2+1}}{\lambda^4\sqrt{2\lambda^2+1}}$$

again decreases monotonically to zero with increasing λ.

The total mean square error H_o is obtained by addition of the distortion term Equation (91) and the noise term Equation (100).

$$H_o = \langle (a-a^*)^2 \rangle_{\substack{ZERO\\DELAY}} = P_a\left[\frac{3\lambda^2+2-2\sqrt{2\lambda^2+1}}{\lambda^2\sqrt{2\lambda^2+1}} + \frac{\mu_n{}^2}{\mu_s}\,\frac{\lambda^4+4\lambda^2+2-(2\lambda^2+2)\sqrt{2\lambda^2+1}}{\lambda^4\sqrt{2\lambda^2+1}}\right] \qquad (102)$$

b) <u>Infinite Delay</u>

(1) Distortion term, D_∞.

$$\mathcal{G}a(t,\infty) = \int_{-\infty}^{\infty} G_\infty(\xi)\,a(t-\xi)\,d\xi \qquad (103)$$

$$a(t) = k\int_0^\infty d\tau\, e^{-k\tau}\alpha(t-\tau)\,d\tau$$

$$\mathcal{G}a(t,\infty) = k\int_{-\infty}^{\infty} d\xi \int_0^\infty d\tau\, G_\infty(\xi)\,e^{-k\tau}\alpha(t-\xi-\tau) \qquad (104)$$

Let

$$\xi + \tau = x \qquad\qquad \tau = x - \xi$$

then x runs from $-\infty$ to ∞ ; ξ runs from $-\infty$ to x.

$$\mathcal{G}a(t,\infty) = k\int_{-\infty}^{\infty} dx \int_{-\infty}^{x} d\xi\, G_\infty(\xi)\,e^{-kx+k\xi}\alpha(t-x) \qquad (105)$$

$$\begin{aligned} \mathcal{G}a(t,\infty) - a(t) &= \int_{-\infty}^{0} dx\left[k\int_{-\infty}^{x} d\xi\, G_\infty(\xi)\,e^{-kx+k\xi}\right]\alpha(t-x) \\ &+ \int_0^\infty dx\left[k\int_{-\infty}^{x} d\xi\, G_\infty(\xi)\,e^{-kx+k\xi} - k e^{-kx}\right]\alpha(t-x) \end{aligned} \qquad (106)$$

Now for $x<0$, we find from Equations (76) and (77)

$$\begin{aligned} k\int_{-\infty}^{x} d\xi\, G_\infty(\xi)\,e^{k\xi-kx} &= \frac{k}{2(C^2-C^{*2})}\int_{-\infty}^{x}\left[C(C^2-k^2)e^{(C+k)\xi-kx} - C^*(C^{*2}-k^2)e^{(C^*+k)\xi-kx}\right]d\xi \\ &= \frac{k}{2(C^2-C^{*2})}\left[C(C-k)e^{Cx} - C^*(C^*-k)e^{C^*x}\right] \end{aligned} \qquad (107)$$

For $x > 0$,

$$k\int_{-\infty}^{x} d\xi G_\infty(\xi) e^{k\xi - kx} = k\int_{-\infty}^{0} d\xi G_\infty(\xi) e^{k\xi - kx} + k\int_{0}^{x} d\xi G_\infty(\xi) e^{k\xi - kx}$$

$$= \frac{k[C^2 - C^{*2} - kC + kC^*] e^{-kx}}{2(C^2 - C^{*2})} + \frac{k}{2(C^2 - C^{*2})}\int_0^x d\xi \left[C(C^2 - k^2) e^{-(C-k)\xi - kx} - C^*(C^{*2} - k^2) e^{-(C^*-k)\xi - kx}\right] \tag{108}$$

$$= \frac{k[C + C^* - k]}{2(C + C^*)} e^{-kx} + \frac{k}{2(C^2 - C^{*2})}\left[C(C+k)(e^{-kx} - e^{-Cx}) - C^*(C^* + k)(e^{-kx} - e^{-C^*x})\right]$$

$$= \left[\frac{k(C + C^* - k)}{2(C + C^*)}\right] e^{-kx} + \frac{k[C^2 - C^{*2} + kC - kC^*]}{2(C^2 - C^{*2})} e^{-kx} - \frac{kC(C+k) e^{-Cx}}{2(C^2 - C^{*2})} + \frac{kC^*(C^*+k) e^{-C^*x}}{2(C^2 - C^{*2})}$$

$$= k e^{-kx} - \frac{kC(C+k)}{2(C^2 - C^{*2})} e^{-Cx} + \frac{kC^*(C^*+k)}{2(C^2 - C^{*2})} e^{-C^*x}$$

From Equations (87), (106), (107) and (108)

$$D_\infty \equiv \langle (g a(t,\infty) - a(t))^2 \rangle = \epsilon_a^2 \int_{-\infty}^{0} dx \left[k\int_{-\infty}^{x} d\xi G_\infty(\xi) e^{-kx + k\xi}\right]^2$$

$$+ \epsilon_a^2 \int_0^\infty dx \left[k\int_{-\infty}^{x} d\xi G_\infty(\xi) e^{-kx + k\xi} - k e^{-kx}\right]^2 \tag{109}$$

$$= \epsilon_a^2 \int_{-\infty}^{0} \frac{k^2}{4(C^2 - C^{*2})^2}\left[C(C-k)e^{Cx} - C^*(C^* - k) e^{C^*x}\right]^2 dx$$

$$+ \epsilon_a^2 \int_0^\infty \frac{k^2}{4(C^2 - C^{*2})^2}\left[C(C+k)e^{-Cx} - C^*(C^*+k) e^{-C^*x}\right]^2 dx$$

$$= \frac{k^2 \epsilon_a^2}{4(C^2 - C^{*2})^2}\int_0^\infty \Big[C^2(C-k)^2 e^{-2Cx} - 2CC^*(C-k)(C^*-k) e^{-(C+C^*)x} + C^{*2}(C^* - k)^2 e^{-2C^*x}$$

$$+ C^2(C+k)^2 e^{-2Cx} - 2CC^*(C+k)(C^*+k) e^{-(C+C^*)x} + C^{*2}(C^*+k)^2 e^{-2C^*x}\Big] dx$$

$$= \frac{k^2 \epsilon_a^2}{4(C^2 - C^{*2})^2}\int_0^\infty \left[2C^2(C^2 + k^2) e^{-2Cx} - 2CC^*(2CC^* + 2k^2) e^{-(C+C^*)x} + 2C^{*2}(C^{*2} + k^2) e^{-2C^*x}\right] dx$$

$$= \frac{k^2 \epsilon_a^2}{4(C^2 - C^{*2})^2}\left[C(C^2 + k^2) - \frac{2CC^*(2CC^* + 2k^2)}{C + C^*} + C^*(C^{*2} + k^2)\right]$$

$$= \frac{k^2 \epsilon_a^2}{4(C^2 - C^{*2})^2}\left[C^3 + C^{*3} - \frac{4C^2 C^{*2}}{C + C^*} + k^2(C + C^*) - \frac{4k^2 CC^*}{C + C^*}\right]$$

$$= \frac{k^2 \epsilon_a^2}{4(C^2 - C^{*2})^2} \left[\frac{C^4 + C^3 C^* - 4C^2 C^{*2} + CC^{*3} + C^{*4}}{C + C^*} + \frac{k^2[(C+C^*)^2 - 4CC^*]}{C + C^*} \right]$$

$$= \frac{k^2 \epsilon_a^2}{4(C^2 - C^{*2})^2} \left[\frac{(C - C^*)^2 (C^2 + 3CC^* + C^{*2})}{C + C^*} + \frac{k^2 (C - C^*)^2}{C + C^*} \right]$$

$$= \frac{k^2 \epsilon_a^2}{4(C + C^*)^3} \left[C^2 + 3CC^* + C^{*2} + k^2 \right]$$

$$= \frac{k^2 \epsilon_a^2 (3CC^* + 2k^2)}{4(C + C^*)^3}$$

or

$$D_\infty = \frac{1}{2} k \epsilon_a^2 \cdot \frac{3\lambda^2 + 2}{2(2\lambda^2 + 1)^{3/2}} = P_a \frac{3\lambda^2 + 2}{2(2\lambda^2 + 1)^{3/2}} \tag{110}$$

$$\sim P_a \frac{3}{4\sqrt{2}} \lambda^{-1} \quad \text{for large } \lambda$$

By comparison with Equation (91) we note that for large λ ,

$$D_\infty = \frac{1}{4} D_0$$

(2) Noise term, N_∞ .

In order to compute the infinite delay noise term N_∞, we replace the upper limit in Equation (93) by t_0 where t_0 will tend to ∞ .

$$\zeta_{t_0}(\tau) = \zeta(\tau, t_0) = \frac{\mu_n \epsilon_a k^2 \lambda^2}{\epsilon_n \sqrt{8}\, \mu_s} \int_{-\infty}^{t_0} dx \int_{-\infty}^{x} dz\, e^{-k|z-\tau|} \eta(x) \tag{111}$$

$$\mathcal{G} \zeta_{t_0}(t, \infty) = \int_{-\infty}^{\infty} d\xi\, G_\infty(\xi) \zeta(t - \xi, t_0) \tag{112}$$

$$= \frac{\mu_n \epsilon_a k^2 \lambda^2}{\epsilon_n \sqrt{8 \mu_s}} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{t_0} dx \int_{-\infty}^{x} dz\, G_\infty(\xi) e^{-k|z - t + \xi|} \eta(x)$$

$$\mathcal{G}\zeta_{t_0}(t,\infty) - \zeta_{t_0}(t) = \frac{\mu_n \epsilon_s k^2 \lambda^2}{\epsilon_n \sqrt{8\mu_s}} \int_{-\infty}^{t_0} dx \left[\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{x} dz\, G_\infty(\xi) e^{-k|z-t+\xi|} - \int_{-\infty}^{x} dz\, e^{-k|z-t|} \right] \eta(x) \qquad (113)$$

Let $z - t = z'$. For $z' < 0$,

$$\int_{-\infty}^{\infty} d\xi\, G_\infty(\xi) e^{-k|z'+\xi|} = \int_{-\infty}^{0} d\xi\, G_\infty(\xi) e^{+k(\xi+z')} + \int_{0}^{-z'} d\xi\, G_\infty(\xi) e^{k(\xi+z')} + \int_{-z'}^{\infty} d\xi\, G_\infty(\xi) e^{-k(\xi+z')}$$

$$= \int_{-\infty}^{0} d\xi \frac{1}{2(C^2 - C^{*2})} \left[C(C^2 - k^2) e^{(C+k)\xi + kz'} - C^*(C^{*2} - k^2) e^{(C^*+k)\xi + kz'} \right]$$

$$+ \int_{0}^{-z'} d\xi \frac{1}{2(C^2 - C^{*2})} \left[C(C^2 - k^2) e^{-(C-k)\xi + kz'} - C^*(C^{*2} - k^2) e^{-(C^*-k)\xi + kz'} \right] \qquad (114)$$

$$+ \int_{-z'}^{\infty} d\xi \frac{1}{2(C^2 - C^{*2})} \left[C(C^2 - k^2) e^{-(C+k)\xi - kz'} - C^*(C^{*2} - k^2) e^{-(C^*+k)\xi - kz'} \right]$$

$$= \frac{1}{2(C^2 - C^{*2})} \Big[C(C-k) e^{kz'} - C^*(C^*-k) e^{kz'} + C(C+k)\left(e^{kz'} - e^{Cz'}\right)$$

$$- C^*(C^*+k)\left(e^{kz'} - e^{C^*z'}\right) + C(C-k) e^{Cz'} - C^*(C^*-k) e^{C^*z'} \Big]$$

$$= \frac{1}{2(C^2 - C^{*2})} \left[(2C^2 - 2C^{*2}) e^{kz'} - 2kCe^{Cz'} + 2kC^* e^{C^*z'} \right]$$

$$= e^{kz'} + \frac{-kCe^{Cz'} + kC^* e^{C^*z'}}{C^2 - C^{*2}}$$

We avoid the need for a separate computation for the case $z' > 0$ by noting

$$F(z') = \int_{-\infty}^{+\infty} d\xi\, G_\infty(\xi) e^{-k|z'+\xi|} = \int_{-\infty}^{+\infty} d\xi\, G_\infty(-\xi) e^{-k|-z'-\xi|}$$

$$= \int_{-\infty}^{+\infty} dx\, G_\infty(x) e^{-k|-z'+x|} = F(-z')$$

so that

$$\int_{-\infty}^{+\infty} d\xi\, G_\infty(\xi) e^{-k|-z'+\xi|} \qquad (115)$$

$$= e^{-k|z'|} + \frac{-kCe^{-C|z'|} + kC^* e^{-C^*|z'|}}{C^2 - C^{*2}}$$

Therefore,

$$\int_{-\infty}^{x} dz\left[\int_{-\infty}^{\infty} d\xi\, G_\infty(\xi)\, e^{-k|z-t+\xi|} - e^{-k|z-t|}\right]$$

$$= \int_{-\infty}^{x-t} dz'\left[\int_{-\infty}^{\infty} d\xi\, G_\infty(\xi)\, e^{-k|z'+\xi|} - e^{-k|z'|}\right] \tag{116}$$

$$= \int_{-\infty}^{x'} dz'\left[\frac{-kCe^{-C|z'|} + kC^*e^{-C^*|z'|}}{C^2-C^{*2}}\right] \qquad (x' = x-t)$$

For $x'<0$, this is

$$\int_{-\infty}^{x'} dz'\left[\frac{-kCe^{Cz'} + kC^*e^{C^*z'}}{C^2-C^{*2}}\right] = \frac{-ke^{Cx'} + ke^{C^*x'}}{C^2-C^{*2}} \tag{117}$$

The integrand in Equation (116) is an even function of z' ; and the integral is zero for $x'=0$; therefore, the integral is an odd function of x' :

$$\int_{-\infty}^{x'} dz'\left[\frac{-kCe^{-C|z'|} + kC^*e^{-C^*|z'|}}{C^2-C^{*2}}\right] = sgn(x')\,\frac{ke^{-C|x'|} - ke^{-C^*|x'|}}{C^2-C^{*2}} \tag{118}$$

Therefore, letting $t_0 - t = t_1$,

$$g\zeta_{t_0}(t,\infty) - \zeta_{t_0}(t) = \frac{\mu_n \epsilon_a k^2 \lambda^2}{\epsilon_n\sqrt{8\mu_s}}\int_{-\infty}^{t_1} dx'\left[\frac{sgn(x')(ke^{-C|x|} - ke^{-C^*|x'|})}{C^2-C^{*2}}\right]\eta(x) \tag{119}$$

and hence,

$$\langle(g\zeta_{t_0}(t,\infty) - \zeta_{t_0}(t))^2\rangle = \frac{\mu_n^2\epsilon_a^2 k^4\lambda^4}{4\mu_s}\int_{-\infty}^{t_1} dx'\,\frac{k^2(e^{-C|x'|} - e^{-C^*|x'|})^2}{(C^2-C^{*2})^2} \tag{120}$$

Letting $t_1 \to \infty$, we obtain

$$
\begin{aligned}
N_\infty &= \frac{\mu_n^2 \epsilon_a^2 k^6 \lambda^4}{4\mu_s} \int_{-\infty}^{\infty} dx' \, \frac{\left(e^{-C|x'|} - e^{-C^*|x'|}\right)^2}{(C^2 - C^{*2})^2} \\
&= \frac{\mu_n^2 \epsilon_a^2 k^2 C^2 C^{*2}}{2\mu_s (C^2 - C^{*2})^2} \int_0^\infty dx \left(e^{-2Cx} - 2e^{-(C+C^*)x} + e^{-2C^*x}\right) \\
&= \frac{\mu_n^2 \epsilon_a^2 k^2 C^2 C^{*2}}{2\mu_s (C^2 - C^{*2})^2} \left[\frac{1}{2C} - \frac{2}{C+C^*} + \frac{1}{2C^*}\right] \\
&= \frac{\mu_n^2 \epsilon_a^2 k^2 C C^*}{4\mu_s (C+C^*)^3 (C-C^*)^2} \left[C^*(C+C^*) - 4CC^* + C(C+C^*)\right] \qquad (121) \\
&= \frac{\mu_n^2 \epsilon_a^2 k^2 C C^*}{4\mu_s (C+C^*)^3} = \frac{1}{2} k \epsilon_a^2 \cdot \frac{\mu_n^2}{\mu_s} \cdot \frac{\lambda^2}{2(2\lambda^2+1)^{3/2}} = P_a \frac{\mu_n^2}{\mu_s} \frac{\lambda^2}{2(2\lambda^2+1)^{3/2}} \\
&\sim P_a \frac{\mu_n^2}{\mu_s} \frac{1}{4\sqrt{2}\,\lambda} \quad \text{for large } \lambda.
\end{aligned}
$$

By comparison with Equation (101) we note that for large λ, $N_\infty = \frac{1}{4} N_0$. The results are summarized below.

By defining the dimensionless parameter $\gamma = \sqrt{2\lambda^2+1}$ our results can be put into a somewhat simpler form for tabulation and comparison. Our results are summarized as follows:

Zero Delay

$$
D_0 = \frac{3\lambda^2 + 2 - 2\sqrt{2\lambda^2+1}}{\lambda^2\sqrt{2\lambda^2+1}} P_a = \frac{3\gamma - 1}{(\gamma+1)\gamma} P_a \qquad (122)
$$

$$
N_0 = \frac{\mu_n^2}{\mu_s} \frac{\lambda^4 + 4\lambda^2 + 2 - (2\lambda^2+2)\sqrt{2\lambda^2+1}}{\lambda^4\sqrt{2\lambda^2+1}} P_a = \frac{\mu_n^2}{\mu_s} \left(\frac{\gamma-1}{\gamma+1}\right)^2 \frac{1}{\gamma} P_a \qquad (123)
$$

$$
H_0 = D_0 + N_0 = \frac{3\gamma^2 + 2\gamma - 1 + \frac{\mu_n^2}{\mu_s}(\gamma-1)^3}{(\gamma+1)^2\gamma} P_a \qquad (124)
$$

For $\quad \dfrac{\mu_n^2}{\mu_s} = 1, \quad H_0 = \dfrac{4\gamma}{(\gamma+1)^2} P_a.$

Infinite Delay

$$D_\infty = \frac{3\lambda^2+2}{2(2\lambda^2+1)^{3/2}} P_a = \frac{3\gamma^2+1}{4\gamma^3} P_a \tag{125}$$

$$N_\infty = \frac{\mu_n^2}{\mu_s} \frac{\lambda^2}{2(2\lambda^2+1)^{3/2}} P_a = \frac{\mu_n^2}{\mu_s} \frac{\gamma^2-1}{4\gamma^3} P_a \tag{126}$$

$$H_\infty = D_\infty + N_\infty = \frac{3\gamma^2+1+\frac{\mu_n^2}{\mu_s}(\gamma^2-1)}{4\gamma^3} P_a \tag{127}$$

For $$\frac{\mu_n^2}{\mu_s} = 1, \quad H_\infty = \frac{1}{\gamma} P_a . \tag{128}$$

It should be born in mind that γ is dependent upon $\mu_s = \frac{\hat{E}_o}{E_o}$ and that the above results are optimum only if $\mu_s = \mu_n = 1$.

4) The Effect of an Initial Phase Error

In an FM system with a stochastic modulation input one intuitively expects that the effects of an initial phase error will not be propagated indefinitely. In fact, for infinite signal-to-noise ratio where the "instantaneous" phase variations can be observed exactly, the initial phase specifies the constant of integration which is equivalent to the occurence of a delta function at the origin ($\tau = t - T$) of the modulating signal. For finite signal-to-noise ratios one then expects the effects of the initial phase to decay at a rate which is determined by the autocorrelation function $R_a(\tau, z) = \frac{k\epsilon_a^2}{2} e^{-k|z-\tau|}$ and the channel quality factor λ .

If in Equation (81) $\Delta\phi \neq 0$ then Equation (82) takes the form

$$a^*(\tau,t) = \frac{\mu_s E_0^2 \beta}{2\epsilon_n^2} \int_{t-T}^{t} \int_{t-T}^{x} R_a(z,\tau)\,dz \left\{ \beta \int_{t-T}^{x} \left[a(u) - a^*(u,t) \right] du + \Delta\phi \right\} dx$$
$$+ \frac{\mu_n E_0 \beta}{2\epsilon_n^2} \int_{t-T}^{t} \int_{t-T}^{x} R_a(z,\tau)\,dz\, \eta(x)\,dx \tag{129}$$

In Equation (129) let $a^*(\tau,t) = a_0^*(\tau,t) + \epsilon(\tau,t)$ where $a_0^*(\tau,t)$ is the coherent solution of Equation (129), i.e., for $\Delta\phi = 0$; and $\epsilon(\tau,t)$ is the variation of $a^*(\tau,t)$ due to $\Delta\phi \neq 0$. So that

$$a_0^*(\tau,t) = \frac{\mu_s E_0^2 \beta}{2\epsilon_n^2} \int_{t-T}^{t} \int_{t-T}^{x} R_a(z,\tau)\,dz \left\{ \beta \int_{t-T}^{x} \left[a(u) - a_0^*(u,t) \right] du \right\} dx$$
$$+ \frac{\mu_n E_0 \beta}{2\epsilon_n^2} \int_{t-T}^{t} \int_{t-T}^{x} R_a(z,\tau)\,dz\, \eta(x)\,dx \tag{130}$$

$$\epsilon(\tau,t) = -\frac{\mu_s E_0^2 \beta}{2\epsilon_n^2} \int_{t-T}^{t} \int_{t-T}^{x} R_a(z,\tau)\,dz \left\{ \beta \int_{t-T}^{x} \epsilon(u,t)\,du - \Delta\phi \right\} dx$$
$$= A \int_{t-T}^{t} h(x,\tau) \int_{t-T}^{x} \epsilon(u,t)\,du\,dx + B \int_{t-T}^{t} h(x,\tau)\,dx \tag{131}$$

where

$$A = -\frac{\mu_s E_0^2 \beta^2}{2\epsilon_n^2} = -\frac{E_0 \hat{E}_0 \beta^2}{2\epsilon_n^2}$$

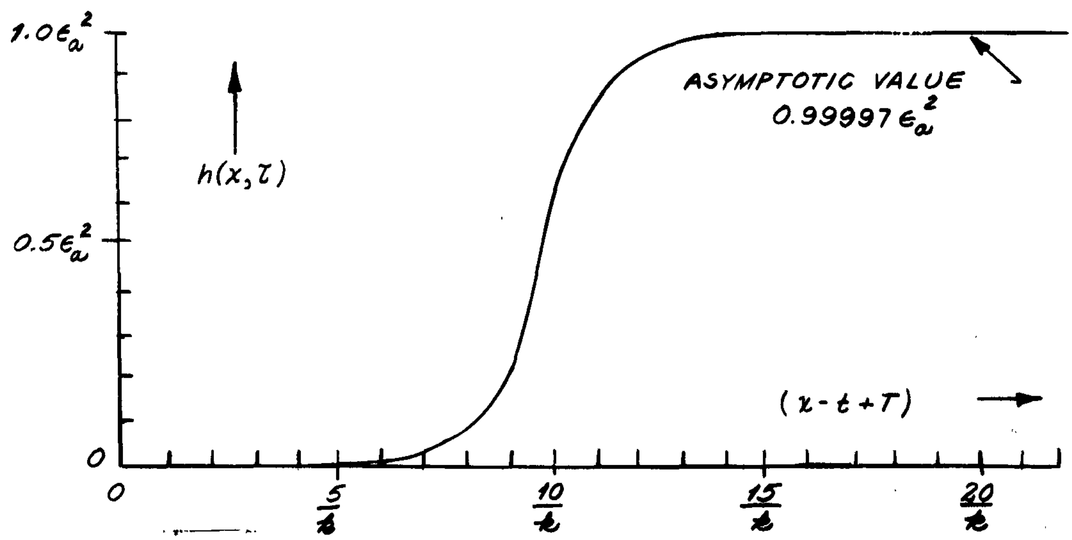
$$B = \frac{\Delta\phi E_0 \hat{E}_0 \beta}{2\epsilon_n^2}$$

$$h(x,\tau) = \int_{t-T}^{x} R_a(z,\tau)\,dz$$

$$= \frac{k\epsilon_a^2}{2}\int_{t-T}^{x} e^{-k|z-\tau|}dz = \begin{cases} \frac{k\epsilon_a^2}{2}\left[\int_{t-T}^{\tau} e^{-k(\tau-z)}dz + \int_{\tau}^{x} e^{-k(z-\tau)}dz\right], & x \geq \tau \\ \frac{k\epsilon_a^2}{2}\int_{t-T}^{x} e^{-k(\tau-z)}dz & , x < \tau \end{cases}$$

$$= \begin{cases} \frac{\epsilon_a^2}{2}\left[2 - e^{-k(\tau-t+T)} - e^{-k(x-\tau)}\right], & x \geq \tau \\ \frac{\epsilon_a^2}{2}\left[e^{-k(\tau-x)} - e^{-k(\tau-t+T)}\right], & x < \tau \end{cases} \tag{132}$$

From the sketch of $h(x,\tau)$, for $\tau - t\,/ = \frac{10}{k}$,



we note that except in a region of width proportional to $1/k$ centered on $x = \tau$,

$$h(x,\tau) \approx \epsilon_a^2 r_o(x-\tau) \tag{133}$$

where the unit step function $r_o(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

We shall here approximate $h(x,\tau)$ by $\epsilon_a^2 r_o(x-\tau)$; this saves a considerable amount of computational work. An exact analysis leading to similar results is presented in Appendix I.

Substituting Equation (133) into Equation (131) yields

$$\epsilon(\tau,t) = A\int_{t-T}^{t} dx\int_{t-T}^{x} \epsilon_a^2 r_o(x-\tau)\epsilon(u,t)\,du + B\int_{t-T}^{t} \epsilon_a^2 r_o(x-\tau)\,dx$$

$$= A\int_{t-T}^{t} du\int_{u}^{t} \epsilon_a^2 r_o(x-\tau)\epsilon(u,t)\,dx + B\epsilon_a^2(t-\tau)$$

$$= A\epsilon_a^2\left\{\int_{t-T}^{\tau}(t-\tau)\epsilon(u,t)\,du + \int_{\tau}^{t}(t-u)\epsilon(u,t)\,du\right\} + B\epsilon_a^2(t-\tau) \tag{134}$$

Equation (134) has solution

$$\epsilon(\tau,t) = -\frac{B}{A}\delta(\tau-t+T) \tag{135}$$

where $\delta(\ )$ is the unit delta function. The occurence of a delta function is due to the approximation of $h(x,\tau)$ by $\epsilon_a^2 r_o(x-\tau)$. The actual solution would also approach a delta function as $k \to \infty$. With k finite one will encounter a transient of duration proportional to $1/k$. In this case $\epsilon(\tau,t) \to 0$ as $\frac{\tau-(t-T)}{k} \gg 1$ and the mean square error is unchanged for such τ .

5) Relationship Between Maximum Likelihood and Conventional FM Reception Above the Threshold

A complete analysis of the behavior of an ideal discriminator (a device the output of which is proportional to the rate of change of phase at its input) has not yet been performed. However, very recently, S. O. Rice has reported the results of his detailed investigation of "Noise in FM Receivers".[7] The analysis presented here was completed before the results of Rice's work became available and does not attempt to treat the behavior of FM receivers near threshold. The present interest is to obtain the limiting performance obtainable with a conventional FM receiver for comparison with the results obtained from the maximum likelihood analysis. Since, in that analysis, it was assumed that the operating conditions are sufficiently good to permit linearization of the governing integral equation, the above threshold behavior of an ideal discriminator is of primary concern. A simplified analysis similar to that reported by W. R. Bennett will be used.[8] The noise spectrum of the discriminator output obtained by our analysis is the first term in an asymptotic expansion of that spectrum, which is valid as the carrier-to-noise ratio becomes very large. The noise output from a discriminator can be obtained by consideration of the statistics of the rate of change of phase of the vector resultant of signal plus noise. The output noise consists essentially of a small gaussian noise current and a succession of impulses or clicks which occur at random whenever the resultant encircles the origin. In the simplified analysis presented below, the possibility of the occurrence of clicks has been eliminated by neglecting the in-phase component of noise. Actually, the rate of occurrence of clicks decreases exponentially with increasing carrier-to-noise ratio.[7] The rapid onset of clicks with decreasing carrier-to-noise ratio reduces the output signal-to-noise ratio, and this is the primary cause of the FM threshold.

In order to obtain results for the FM receiver comparable with those obtained for the case of maximum likelihood estimation, we will assume that the channel conditions are so good that operation is above the threshold.

The use of zero delay and infinite delay least mean square error (Wiener) filters connected to the discriminator output will yield results for comparison with zero and infinite delay maximum likelihood estimation.

The design of the Wiener filter requires specification of the power spectra (or equivalent) of the signal and noise components at the input to the filter*. An approximation, valid for high carrier-to-noise ratios, of the discriminator output noise spectrum due to white gaussian RF noise may be obtained as follows.[3] Setting $a(\tau)=0$ in Equation (1), the received signal is

$$e_1(\tau) = E_0 \sin \omega_0 \tau + n(\tau) \tag{136}$$

which may be written as

$$\begin{aligned} e_1(\tau) &= E_0 \sin \omega_0 \tau + n_s(\tau) \sin \omega_0 \tau + n_c(\tau) \cos \omega_0 \tau \\ &= \left[E_0 + n_s(\tau)\right] \sin \omega_0 \tau + n_c(\tau) \cos \omega_0 \tau \\ &= A(\tau) \sin\left[\omega_0 \tau + \psi(\tau)\right] \end{aligned} \tag{137}$$

where

$$\begin{aligned} A^2(\tau) &= \left[E_0 + n_s(\tau)\right]^2 + n_c^2(\tau) \\ \psi(\tau) &= \tan^{-1} \frac{n_c(\tau)}{E_0 + n_s(\tau)} \end{aligned} \tag{138}$$

The discriminator output is proportional to

$$\dot{\psi}(\tau) = \frac{\left[E_0 + n_s(\tau)\right] \dot{n}_c(\tau) - n_c(\tau) \cdot \dot{n}_s(\tau)}{\left[E_0 + n_s(\tau)\right]^2 + n_c^2(\tau)} \approx \frac{\dot{n}_c(\tau)}{E_0} \tag{139}$$

*The results of the analysis are summarized on page 50 .

for large carrier-to-noise ratios.

The spectral density of $\dot{\psi}(\tau)$ due to noise is then obtained as

$$W_{\dot{\psi}}(\omega) = \frac{\omega^2}{E_o^2} W_{n_c}(\omega) = \frac{2\epsilon_n^2 \omega^2}{E_o^2} \tag{140}$$

The discriminator gain is determined as $1/\beta$ by the requirement that the output in the absence of noise be $a(\tau)$.

Combining these results we find that the signal spectrum $S_a(\omega)$ and noise spectrum $N_D(\omega)$ at the discriminator output above threshold are given by

$$S_a(\omega) = \epsilon_a^2 \frac{k^2}{k^2+\omega^2} \text{ watts/cps (double-sided spectrum)} \tag{141}$$

$$N_D(\omega) = \frac{2\epsilon_n^2 \omega^2}{\beta^2 E_o^2} \text{ watts/cps (double-sided spectrum)} \tag{142}$$

The total signal power is

$$P_a = \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_a^2 \frac{k^2}{k^2+\omega^2} d\omega = \frac{k}{2} \epsilon_a^2 \tag{143}$$

The least mean square (Wiener) filter can now be designed on the basis of Equation (144) for the infinite delay or Equation (151) for the zero delay case. An excellent exposition of the theory of least mean square filtering is presented by Bode and Shannon in Reference [9]. The pertinent results of that paper are abstracted below.

Assuming stationary statistics for both signal and additive noise, the transfer function of the least mean square infinite delay filter is given by

$$Y(\omega) = \frac{S(\omega)}{S(\omega)+N(\omega)} \tag{144}$$

where $S(\omega)$ is the spectral density of the signal and $N(\omega)$ is the spectral density of the noise. The total mean square error resulting from use of this filter is*

$$H_\infty = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{S(\omega)\,N(\omega)}{S(\omega)+N(\omega)}\,d\omega \tag{145}$$

The least mean square zero delay filter $Y_4(\omega)$ is obtained as follows. Let

$$Y_1(\omega)\,Y_1^*(\omega) = \frac{1}{S(\omega)+N(\omega)} \tag{146}$$

where $Y_1(\omega)$ is a realizable frequency response with all of its singularities in the upper half of the ω plane.

$$Y_2(\omega) = Y_1^{-1}(\omega)\,Y(\omega) = \frac{S(\omega)}{S(\omega)+N(\omega)}\,Y_1^{-1}(\omega) \tag{147}$$

$$k_2(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y_2(\omega)\,e^{i\omega t}\,d\omega \quad \text{, impulse response corresponding to } Y_2(\omega) \tag{148}$$

$$k_3(t) = \begin{cases} k_2(t)\,, & t \geq 0 \\ 0\,, & t < 0 \end{cases} \quad \text{, realizable part of } k_2(t) \tag{149}$$

$$Y_3(\omega) = \int_0^{\infty} k_3(t)\,e^{-i\omega t}\,dt \quad \text{, frequency response corresponding to } k_3(t) \tag{150}$$

Then

$$Y_4(\omega) = Y_3(\omega)\,Y_1(\omega)\,, \tag{151}$$

and the total mean square error H_0 resulting from use of this filter is given by

$$H_0 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\{ S(\omega)\,|1 - Y_4(\omega)|^2 + N(\omega)\,|Y_4(\omega)|^2 \right\} d\omega \tag{152}$$

*The factor $\frac{1}{2\pi}$ does not appear in Reference [9] because the spectral density used there is on a per radian basis, whereas a per cps basis has been used in this paper.

We shall again be interested in the performance of this system under off-design conditions. As before, we will evaluate the deterioration when the system is designed on the assumption of RF noise power density ϵ_n^2 and carrier amplitude E_o and actually encounters $\hat{\epsilon}_n^2 = \mu_n^2 \epsilon_n^2$ and $\hat{E}_o = \mu_s E_o$. The actual noise spectrum of the discriminator output is then $\hat{N}_D(\omega) = \frac{\mu_n^2}{\mu_s^2} N_D(\omega)$.

The performance of this system with actual discriminator noise output $\hat{N}_D(\omega)$ can then be obtained by first assuming $\mu_s = \mu_n = 1$, computing $N_\infty, D_\infty, N_o, D_o$ and then setting

$$\hat{N}_\infty = \frac{\mu_n^2}{\mu_s^2} N_\infty, \quad \hat{N}_o = \frac{\mu_n^2}{\mu_s^2} N_o \qquad (153)$$

$$\hat{D}_\infty = D_\infty \quad , \quad \hat{D}_o = D_o$$

where the circumflexes denote actual rather than design conditions, and N_∞, D_∞, N_o, D_o are the contributions to the mean square error of the infinite and zero delay filters due to noise and distortion respectively.

The channel quality factor λ was previously (Equation (83)) defined as

$$\lambda = \left[\frac{\mu_s E_o^2 \beta^2 \epsilon_a^2}{2 k^2 \epsilon_n^2} \right]^{1/4} \qquad (154)$$

Let

$$\lambda_o = \left[\frac{E_o^2 \beta^2 \epsilon_a^2}{2 k^2 \epsilon_n^2} \right]^{1/4} \qquad (155)$$

i.e., λ_o is the value λ assumes when $\mu_s = \frac{\hat{E}_o}{E_o} = 1$. The nominal discriminator output noise spectrum, Equation (142), is then given by

$$N_D(\omega) = \frac{\epsilon_a^2 \omega^2}{k^2 \lambda_o^4} \qquad (156)$$

a) <u>Infinite Delay Case</u>

Substituting Equations (141) and (156) into Equation (144), the frequency response of the filter is

$$Y(\omega)=\frac{S_a(\omega)}{S_a(\omega)+N_0(\omega)}=\frac{\dfrac{\epsilon_a^2 k^2}{k^2+\omega^2}}{\dfrac{\epsilon_a^2 k^2}{k^2+\omega^2}+\dfrac{\epsilon_a^2\omega^2}{k^2\lambda_0^4}} \tag{157}$$

$$=\frac{(k\lambda_0)^4}{\omega^4+k^2\omega^2+(k\lambda_0)^4}$$

With this filter the mean square difference between the filter output and the signal component of the input is given by*

$$H_\infty=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{S_a(\omega)\,N_0(\omega)}{S_a(\omega)+N(\omega)}\,d\omega=\frac{\epsilon_a^2 k^2}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega^2}{\omega^4+k^2\omega^2+(k\lambda_0)^4}\,d\omega$$

$$=\frac{\epsilon_a^2 k^2}{2\pi}I_1=\frac{1}{\gamma_0}\,\frac{k}{2}\,\epsilon_a^2=\frac{1}{\gamma_0}P_a \tag{158}$$

where

$$\gamma_0\equiv\sqrt{2\lambda_0^2+1}\sim\sqrt{2}\,\lambda_0 \qquad \text{for} \qquad \lambda_0\gg 1 \tag{159}$$

so that

$$H_\infty\sim\frac{1}{\sqrt{2}\,\lambda_0}P_a \qquad \text{for} \qquad \lambda_0\gg 1.$$

The mean square error consists of two independent components, $H_\infty=N_\infty+D_\infty$. Where N_∞ denotes the mean square output due to noise and D_∞ the mean square distortion of the signal as has already been mentioned.

---

*The integral $I_1=\int_{-\infty}^{+\infty}\frac{\omega^2}{\omega^4+k^2\omega^2+(k\lambda_0)^4}d\omega$ and the integrals I_2, I_3, I_4, and I_5 which will be encountered later are evaluated in Appendix II.

$$N_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} N_D(\omega)\,|\Upsilon(\omega)|^2 d\omega$$

$$= \frac{\epsilon_a^2 k^6 \lambda_0^4}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2}{(\omega^4 + k^2\omega^2 + k^4\lambda_0^4)^2}\, d\omega$$

$$= \frac{\epsilon_a^2 k^6 \lambda_0^4}{2\pi} I_5 = \frac{\gamma_0^2 - 1}{4\gamma_0^3}\,\frac{k}{2}\,\epsilon_a^2 = \frac{\gamma_0^2-1}{4\gamma_0^3} P_a \tag{160}$$

$$N_\infty \sim \frac{1}{4\sqrt{2}\,\lambda_0} P_a \qquad \text{for} \quad \lambda_0 \gg 1. \tag{161}$$

$$D_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega)\,|1 - \Upsilon(\omega)|^2 d\omega$$

$$= H_\infty - N_\infty = \frac{3\gamma_0^2 + 1}{4\gamma_0^3} P_a \tag{162}$$

$$D_\infty \sim \frac{3}{4\sqrt{2}\,\lambda_0} P_a \qquad \text{for } \lambda_0 \gg 1. \tag{163}$$

b) <u>Zero Delay Case</u>

The transfer function $Y_4(\omega)$ of the zero delay Wiener filter will now be computed, in accordance with the procedure previously outlined.

$$Y_1(\omega)\, Y_1^*(\omega) = \frac{1}{S_a(\omega) + N_D(\omega)} = \frac{1}{\epsilon_a^2 \frac{k^2}{\omega^2 + k^2} + \epsilon_a^2 \frac{1}{k^2 \lambda_0^4}\omega^2} \tag{164}$$

$$= \frac{k^2\lambda_0^4}{\epsilon_a^2}\,\frac{\omega^2 + k^2}{\omega^4 + k^2\omega^2 + k^4\lambda_0^4}$$

Let

$$b_1 \equiv \frac{k}{2}\sqrt{2\lambda_0^2 + 1} \tag{165}$$

$$b_2 \equiv \frac{k}{2}\sqrt{2\lambda_0^2 - 1} \tag{166}$$

Then $\quad b_1^2 + b_2^2 = k^2\lambda_0^2 \quad$ and $\quad 2(b_1^2 + b_2^2) - 4b_2^2 = k^2 \quad$ so that

$$\omega^4 + k^2\omega^2 + k^4\lambda_0^4 = \left\{\omega^2 + (b_1^2 + b_2^2)\right\}^2 - (2b_2\omega)^2$$

$$= \left\{\omega^2 + (b_1^2 + b_2^2) - 2b_2\omega\right\}\left\{\omega^2 + (b_1^2 + b_2^2) + 2b_2\omega\right\}$$

$$= \left\{(\omega - b_2)^2 + b_1^2\right\}\left\{(\omega + b_2)^2 + b_1^2\right\}$$

$$= (\omega - b_2 - ib_1)(\omega - b_2 + ib_1)(\omega + b_2 - ib_1)(\omega + b_2 + ib_1)$$

Hence, we write

$$\frac{1}{S_a(\omega) + N_D(\omega)} = \left\{\frac{k\lambda_0^2}{\epsilon_a}\,\frac{-i(\omega - ik)}{(\omega - b_2 - ib_1)(\omega + b_2 - ib_1)}\right\}$$

$$\cdot\left\{\frac{k\lambda_0^2}{\epsilon_a}\,\frac{i(\omega + ik)}{(\omega - b_2 + ib_1)(\omega + b_2 + ib_1)}\right\}$$

$$= \left\{\Upsilon_1(\omega)\right\}\cdot\left\{\Upsilon_1^*(\omega)\right\} \tag{167}$$

$$\Upsilon_1(\omega) = \frac{k\lambda_0^2}{\epsilon_a}\,\frac{-i(\omega - ik)}{(\omega - b_2 - ib_1)(\omega + b_2 - ib_1)} \tag{168}$$

and

$$\Upsilon_2(\omega) = \frac{S_a(\omega)}{S_a(\omega) + N_D(\omega)}\,\frac{1}{\Upsilon_1(\omega)}$$

$$= \epsilon_a k^3\lambda_0^2\,\frac{i}{(\omega - ik)(\omega - b_2 + ib_1)(\omega + b_2 + ib_1)}$$

$$= i\epsilon_a k^3\lambda_0^2\left(\frac{A_1}{\omega - ik} + \frac{A_2}{\omega - b_2 + ib_1} + \frac{A_3}{\omega + b_2 + ib_1}\right) \tag{169}$$

where

$$A_1 = \frac{-1}{(b_2 - ib_1 - ik)(b_2 + ib_1 + ik)} = -\frac{1}{k^2\lambda_0^2}\,\frac{\gamma_0 - 1}{\gamma_0 + 1}$$

$$A_2 = \frac{1}{2b_2(b_2 - ib_1 - ik)} = \frac{1}{k^2\lambda_0^2}\,\frac{\gamma_0 - 1}{\gamma_0 + 1}\,\frac{b_2 + ib_1 + ik}{2b_2}$$

$$A_3 = \frac{1}{2b_2(b_2 + ib_1 + ik)} = \frac{1}{k^2\lambda_0^2}\,\frac{\gamma_0 - 1}{\gamma_0 + 1}\,\frac{b_2 - ib_1 - ik}{2b_2}.$$

The impulse response corresponding to $Y_2(\omega)$ is

$$\begin{aligned} k_2(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y_2(\omega)\, e^{i\omega t} d\omega \\ &= i\,\frac{\epsilon_a k^3 \lambda_o^2}{2\pi}\int_{-\infty}^{\infty}\left(\frac{A_1}{\omega - ik} + \frac{A_2}{\omega - b_2 + ib_1} + \frac{A_3}{\omega + b_2 + ib_1}\right) e^{i\omega t} d\omega \end{aligned} \tag{170}$$

Since $\displaystyle\int_0^{\infty} (ie^{-kt})\, e^{-i\omega t} dt = \frac{1}{\omega - ik}$ and $\displaystyle\int_{-\infty}^{0} (-ie^{iat})\, e^{-i\omega t} dt = \frac{1}{\omega - \alpha}$

for $\alpha = b_2 - ib_1$ or $\alpha = -b_2 - ib_1$,

the impulse response $k_2(t)$ is:

For $t \geq 0$,

$$\begin{aligned} k_2(t) &= i\epsilon_a k^3 \lambda_o^2 A_1 (ie^{-kt}) \\ &= \frac{\gamma_o - 1}{\gamma_o + 1}\, \epsilon_a k e^{-kt} \end{aligned} \tag{171}$$

For $t < 0$,

$$\begin{aligned} k_2(t) &= i\epsilon_a k^3 \lambda_o^2 \left\{ A_2\left(-ie^{i(b_2 - ib_1)t}\right) + A_3\left(-ie^{i(-b_2 - ib_1)t}\right)\right\} \\ &= \frac{\gamma_o - 1}{\gamma_o + 1}\, \epsilon_a k\, \frac{e^{-b_1|t|}}{2b_2}\Big\{ (b_2 + ib_1 + ik)e^{ib_2 t} \\ &\qquad + (b_2 - ib_1 - ik)e^{-ib_2 t}\Big\} \end{aligned} \tag{172}$$

Actually, we do not need $k_2(t)$ for $t > 0$, because the realizable frequency response $Y_3(\omega)$ is obtained from the impulse response

$$k_3(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \dfrac{\gamma_o - 1}{\gamma_o + 1}\, \epsilon_a k e^{-kt} & \text{for} \quad t \geq 0 \end{cases} \tag{173}$$

So
$$\Upsilon_3(\omega) = \frac{\gamma_0 - 1}{\gamma_0 + 1} \epsilon_a k \int_0^\infty e^{-kt} e^{-i\omega t} dt$$

$$= \frac{\gamma_0 - 1}{\gamma_0 + 1} \epsilon_a \frac{-ik}{\omega - ik} \tag{174}$$

Finally, the frequency response of the required filter is

$$\Upsilon_4(\omega) = \Upsilon_3(\omega)\Upsilon_1(\omega)$$

$$= -\frac{\gamma_0 - 1}{\gamma_0 + 1} \frac{k^2 \lambda_0^2}{(\omega - b_2 - ib_1)(\omega + b_2 - ib_1)}$$

$$= \frac{\gamma_0 - 1}{\gamma_0 + 1} \frac{k^2 \lambda_0^2}{-\omega^2 + 2b_1 i\omega + k^2 \lambda_0^2} \tag{175}$$

The mean square error resulting from use of this filter can now be determined. The mean square error due to the noise

$$N_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} N_D(\omega) \left| \Upsilon_4(\omega) \right|^2 d\omega$$

$$= \frac{1}{2\pi} \frac{\epsilon_a^2}{k^2 \lambda_0^4} \left(\frac{\gamma_0 - 1}{\gamma_0 + 1} \right)^2 \int_{-\infty}^{\infty} \frac{\omega^2 k^4 \lambda_0^4}{(k^2 \lambda_0^2 - \omega^2)^2 + 4b_1^2 \omega^2} d\omega$$

$$= \frac{1}{2\pi} \epsilon_a^2 k^2 \left(\frac{\gamma_0 - 1}{\gamma_0 + 1} \right)^2 \int_{-\infty}^{\infty} \frac{\omega^2}{\omega^4 + k^2 \omega^2 + k^4 \lambda_0^4} d\omega$$

$$= \frac{k^2}{2\pi} \epsilon_a^2 \left(\frac{\gamma_0 - 1}{\gamma_0 + 1} \right)^2 I_1$$

$$= \left(\frac{\gamma_0 - 1}{\gamma_0 + 1} \right)^2 \frac{1}{\gamma_0} \frac{k}{2} \epsilon_a^2 = \left(\frac{\gamma_0 - 1}{\gamma_0 + 1} \right)^2 \frac{1}{\gamma_0} P_a \tag{176}$$

$$N_0 \sim \frac{1}{\sqrt{2}\,\lambda_0} P_a \qquad \text{for} \quad \lambda_0 \gg 1.$$

The mean square error due to distortion of the signal

$$D_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_a(\omega) \left| 1 - \Upsilon_4(\omega) \right|^2 d\omega$$

$$= \frac{1}{2\pi} \epsilon_a^2 k^2 \int_{-\infty}^{\infty} \frac{1}{\omega^2 + k^2} \left| 1 - \Upsilon_4(\omega) \right|^2 d\omega \tag{177}$$

Since $$\left|1-Y_4(\omega)\right|^2 = 1+\left|Y_4(\omega)\right|^2 - \left[Y_4(\omega)+Y_4^*(\omega)\right]$$

$$= 1+\left(\frac{\gamma_0-1}{\gamma_0+1}\right)^2 \frac{k^4\lambda_0^4}{\omega^4+k^2\omega^2+k^4\lambda_0^4}$$

$$- \frac{\gamma_0-1}{\gamma_0+1} k^2\lambda_0^2 \frac{2(-\omega^2+k^2\lambda_0^2)}{\omega^4+k^2\omega^2+k^4\lambda_0^4}$$

$$= 1+k^2(\gamma_0-1)^2 \frac{\omega^2}{\omega^4+k^2\omega^2+k^4\lambda_0^4}$$

$$-k^4\frac{(\gamma_0-1)^3}{4}(\gamma_0+3)\frac{1}{\omega^4+k^2\omega^2+k^4\lambda_0^4}$$

and $$\int_{-\infty}^{\infty}\frac{1}{\omega^2+k^2}\,d\omega = \frac{\pi}{k},$$

$$D_0 = \frac{1}{2\pi}\epsilon_a^2 k^2\left\{\frac{\pi}{k}+k^2(\gamma_0-1)^2 I_3 - k^4\frac{(\gamma_0-1)^3}{4}(\gamma_0+3)I_4\right\}$$

$$= \frac{1}{2}k\epsilon_a^2\left\{1+\frac{2}{\gamma_0}\left(\frac{\gamma_0-1}{\gamma_0+1}\right)^2-\frac{1}{\gamma_0}\left(\frac{\gamma_0-1}{\gamma_0+1}\right)^2(\gamma_0+3)\right\}$$

$$= \frac{1}{2}k\epsilon_a^2\left\{1-\frac{(\gamma_0-1)^2}{\gamma_0(\gamma_0+1)}\right\}$$

$$= \frac{3\gamma_0-1}{\gamma_0+1}\,\frac{1}{\gamma_0}\,P_a \tag{178}$$

$$D_0 \sim \frac{3}{\sqrt{2}\,\lambda_0}P_a \qquad \text{for} \quad \lambda_0 \gg 1. \tag{179}$$

The total mean square error

$$H_0 = N_0 + D_0 = \frac{4\gamma_0}{(\gamma_0+1)^2}P_a \tag{180}$$

$$H_0 \sim \frac{4}{\sqrt{2}\,\lambda_0}P_a \qquad \text{for} \quad \lambda_0 \gg 1. \tag{181}$$

We note that

$$\frac{H_0}{H_\infty} = \left(\frac{2\gamma_0}{\gamma_0+1}\right)^2 \sim 4 \qquad \text{for} \quad \lambda_0 \gg 1. \tag{182}$$

$$\frac{N_o}{N_\infty} = \frac{4\gamma_o^2(\gamma_o - 1)}{(\gamma_o + 1)^3} \sim 4 \qquad \text{for} \qquad \lambda_o \gg 1. \tag{183}$$

$$\frac{D_o}{D_\infty} = \frac{4\gamma_o^2(3\gamma_o - 1)}{(\gamma_o + 1)(3\gamma_o^2 + 1)} \sim 4 \qquad \text{for} \qquad \lambda_o \gg 1. \tag{184}$$

c) <u>Off-Design Performance</u>

Assuming that deviations of the carrier and noise levels are such that the assumption of a large carrier-to-noise power ratio remains valid, the effects of such deviations are easily taken into account. At the output of the discriminator the noise power spectrum is now $\hat{N}_D(\omega) = \left(\frac{\mu_n}{\mu_s}\right)^2 N_D(\omega)$ while the signal spectrum $S(\omega)$ remains unchanged. Therefore, at the output of the filter we have

$$\begin{aligned} \hat{N}_i &= \left(\frac{\mu_n}{\mu_s}\right)^2 N_i \\ \hat{D}_i &= D_i \qquad\qquad\qquad (i = 0, \infty) \\ \hat{H}_i &= H_i + \left[\left(\frac{\mu_n}{\mu_s}\right)^2 - 1\right] N_i \end{aligned} \tag{185}$$

where the circumflex denotes off-design conditions.

The results obtained for the discriminator Wiener-filter receiver are summarized below.

Zero Delay:

$$\hat{D}_o = \frac{3\gamma_o - 1}{(\gamma_o + 1)\gamma_o} P_a = D_o \tag{186}$$

$$\hat{N}_o = \frac{\mu_n^2}{\mu_s^2}\left(\frac{\gamma_o - 1}{\gamma_o + 1}\right)^2 \frac{1}{\gamma_o} P_a \tag{187}$$

$$\hat{H}_0 = \frac{3\gamma_0^2 + 2\gamma_0 - 1 + \frac{\mu_n^2}{\mu_s^2}(\gamma_0 - 1)^2}{(\gamma_0 + 1)^2} P_a \tag{188}$$

$$\left(\text{For } \frac{\mu_n}{\mu_s} = 1, \quad \hat{H}_0 = \frac{4\gamma_0}{(\gamma_0+1)^2} P_a = H_0\right) \tag{189}$$

Infinite Delay:

$$\hat{D}_\infty = \frac{3\gamma_0^2 + 1}{4\gamma_0^3} P_a = D_\infty \tag{190}$$

$$\hat{N}_\infty = \frac{\mu_n^2}{\mu_s^2} \frac{\gamma_0^2 - 1}{4\gamma_0^3} P_a \tag{191}$$

$$\hat{H}_\infty = \frac{3\gamma_0^2 + 1 + \frac{\mu_n^2}{\mu_s^2}(\gamma_0^2 - 1)}{4\gamma_0^3} P_a \tag{192}$$

$$\left(\text{For } \frac{\mu_n}{\mu_s} = 1, \quad \hat{H}_\infty = \frac{1}{\gamma_0} P_a = H_\infty\right) \tag{193}$$

We note that when $\mu_s = 1$ we have $\gamma = \gamma_0$, and these expressions are identical with those obtained for the case of maximum likelihood estimation, Equations (122) to (128).

DISCUSSION OF RESULTS

The most striking result obtained is the complete agreement of the six expressions describing the mean square error H and its decomposition into distortion, D, and noise, N, terms obtained by the two different analyses (maximum likelihood estimation and demodulation by means of an ideal discriminator followed by a Wiener filter) for the case when operation is under the assumed design conditions. Since all the results were obtained by the use of approximations valid only when operating with a high carrier-to-noise ratio it is desirable to obtain an estimate of the range of reasonable validity of these results. Such an estimate can be easily obtained for the discriminator-Wiener filter case by examining the approximation made in the derivation of the results.

This approximation is contained in Equation (139) where we set*

$$\frac{[E_0 + n_s(\tau)]\,\dot{n}_c(\tau) - n_c(\tau)\,\dot{n}_s(\tau)}{[E_0 + n_s(\tau)]^2 + n_c^2(\tau)} \approx \frac{\dot{n}_c(\tau)}{E_0} \tag{194}$$

This requires that $E_0^2 \gg \langle n_c^2(\tau) \rangle$ and, therefore, it is essential that the bandwidth of the input white noise be limited. In practice, the noise power is limited by the receiver i.f. bandwidth. The required i.f. bandwidth $B_{i.f.}$ is given approximately by

$$B_{i.f.} \cong 2A\frac{\beta}{2\pi} + \frac{k}{2} \quad cps \tag{195}$$

where

A · · · maximum signal amplitude

$\frac{k}{2}$ · · · noise equivalent signal bandwidth (two-sided spectrum)

*This approximation is good only "most of the time" since it obviously does not hold when $\dot{n}_c(\tau) \to 0$. Reference 7 considers this problem in detail.

Since $a(\tau)$ is gaussianly distributed, a maximum signal amplitude A cannot be rigorously specified. However, the probability $|a(\tau)| > A$ is given by

$$P\left\{|a(\tau)| > A\right\} = 1 - \operatorname{erf}\left(\frac{A}{\sqrt{2P_a}}\right) \tag{196}$$

where P_a, the modulation power, is also the expected value of $a^2(\tau)$; $P_a = \langle a^2(\tau)\rangle$. If we choose $A/\sqrt{2P_A} = 4$ the probability that $|a(\tau)| > A = 4\sqrt{2P_A}$ is less than 10^{-7}. The required i.f. bandwidth $B_{i.f.}$ is then

$$B_{i.f.} = \frac{\sqrt{32P_a}\,\beta}{\pi} + \frac{k}{2} \quad \text{cps (one-sided spectrum)} \tag{197}$$

Since $\langle n_c^2\rangle = 2\epsilon_n^2 B_{i.f.}$ = Noise Power in i.f. bandwidth, and $E_o^2/2$ = Carrier Power, the r.f. carrier-to-noise power ratio

$$\rho = \frac{E_o^2/2}{2\epsilon_n^2 B_{i.f.}} = \frac{E_o^2}{2\langle n_c^2\rangle} \tag{198}$$

Assuming that the approximation in Equation (194) becomes valid for $\frac{E_o^2}{\langle n_c^2\rangle} \geq 20$ the required r.f. signal-to-noise power ratio $\rho \geq$ 10db. The above relations can now be used to determine the approximate minimum value of the channel quality factor λ_o required for our results to be valid.

Using Equations (198) and (197) we have

$$\rho = \frac{E_o^2}{4\epsilon_n^2 B_{i.f.}} = \frac{E_o^2}{4\epsilon_n^2}\,\frac{1}{\frac{\sqrt{32P_a}\,\beta}{\pi} + \frac{k}{2}} \geq 10 \tag{199}$$

The design channel quality factor λ_o may be expressed in various forms. Thus, starting with Equation (155)

$$\lambda_o^4 = \frac{\beta^2\epsilon_a^2 E_o^2}{2k^2\epsilon_n^2} = 2\frac{\beta^2}{k^2}\epsilon_a^2\left(\frac{E_o^2}{4\epsilon_n^2}\right)$$

by substituting the expressions for P_a, ρ and $B_{i.f.}$ one obtains

$$\lambda_o^4 = 4\left(\frac{\beta^2}{k^2}P_a\right)\left(\frac{B_{i.f.}}{k}\right)\rho$$

$$= 4\left(\frac{\beta^2}{k^2}P_a\right)\left[\frac{\sqrt{32}}{\pi}\left(\frac{\beta^2}{k^2}P_a\right)^{1/2} + \frac{1}{2}\right]\rho \tag{200}$$

Note that each of the terms in Equation (200) is a dimensionless parameter. By substituting $\rho \geq 10$ in Equation (200) the minimum required λ_o for applicability of our results is determined as

$$\lambda_o^4 \geq 40\left(\frac{\beta^2}{k^2}P_a\right)\left(\frac{B_{i.f.}}{k}\right)$$

$$= 40\left(\frac{\beta^2}{k^2}P_a\right)\left[\frac{\sqrt{32}}{\pi}\left(\frac{\beta^2}{k^2}P_a\right)^{1/2} + \frac{1}{2}\right] \tag{201}$$

It is especially to be noted that the minimum value of λ_o required for above threshold operation cannot be specified without considering the numerical values of the parameters appearing in Equation (200).

Similarly, the range of applicability of the results of the analysis of maximum likelihood estimation is restricted by the linearizing assumption that $\sin\beta\int_{t-T}^{x}[a(u)-a^*(u,t)]\,du$ can be replaced by $\beta\int_{t-T}^{x}[a(u)-a^*(u,t)]\,du$. It has not yet been determined what combination of parameters are required to justify making this assumption.

Figure 1 illustrates the performance of both the maximum likelihood and the Discriminator-Wiener filter receivers operating under design conditions. Plots of $\frac{P_a}{H_o}$, $\frac{P_a}{D_o}$, $\frac{P_a}{N_o}$, $\frac{P_a}{H_\infty}$, $\frac{P_a}{D_\infty}$ and $\frac{P_a}{N_\infty}$ are presented in accordance with Equations (124), (122), (123), (127), (125) and (126). The scale of ordinates expresses the above ratios in decibels, and the scale of abscissas expresses λ_o^4, the fourth power of the channel quality factor, also in decibels. It is important to bear in mind that these demodulation systems were designed so as to minimize H_o, H_∞ and, hence, maximize

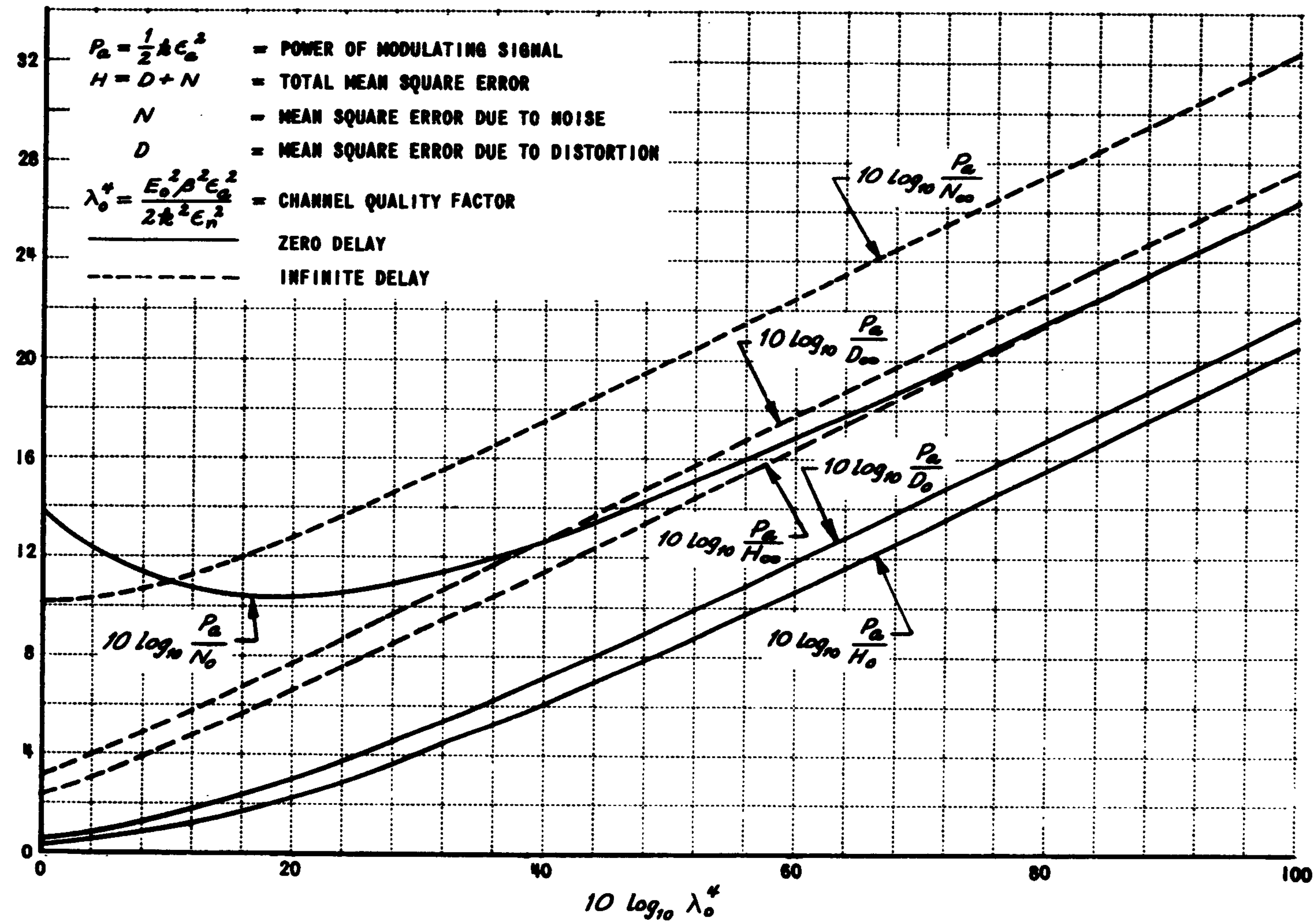


Figure I PERFORMANCE OF MAXIMUM LIKELIHOOD OR DISCRIMINATOR-WIENER FILTER FM RECEIVERS OPERATING UNDER DESIGN CONDITIONS

P_a/H and that no attempt has been made to control the division of H into its component terms, D and N. From this figure it is seen that for large λ_0 all curves approach a slope of $\frac{1}{4}$, this reflects the fact that H, D, N all are of order $1/\lambda_0$. It will also be noted that distortion accounts for the major portion of the mean square error. For large values of λ_0, 75% of H is due to distortion and this percentage increases as λ_0 is decreased. Figures 2, 3, 4 and 5 describe in various ways the effects of operating conditions differing from the design conditions. From Equations (122) - (128) and (186) - (193) it is seen that the variation of the mean square error and its components, due to deviations from design conditions, is not the same for maximum likelihood estimation and demodulation by a discriminator followed by a Wiener filter. Off-design operation can be due to encountering noise and/or carrier strength other than anticipated, $\mu_n \neq 1$ and/or $\mu_s \neq 1$. Since the mean square error H is a function of three variables, λ_0, μ_n, μ_s, a complete graphical presentation is not practical. In Figure 2 a design value of $\lambda_0 = 10$ is assumed, and the effect of varying the received carrier strength $\hat{E}_0 = \mu_s E_0$ is displayed. It will be noted that while an increase in carrier strength above the design value does improve the performance, the improvement is not as great as if the receiver had been designed for this value of carrier strength. For example, for $\lambda_0^4 = 40$db we find from Figure 1 or Figure 2 that $P_a/H_0 = 6.1$db. If the carrier strength is now increased by 10db and the receiver design not adjusted, then Figure 2 shows $P_a/H_0 = 7.0$db for the Discriminator-Wiener filter and $P_a/H_0 = 7.9$db for maximum likelihood estimation. From Figure 1 we find that if the receivers had been designed for this condition, $\lambda_0^4 = 50$db, $P_a/H_0 = 8.3$db, with either system of demodulation.

In Figure 3 the change in mean square error due to noise and distortion expressed in db is plotted for the Discriminator-Wiener filter receivers. From Equations (186), (187), (190) and (191) it is seen that this is a function of μ_n/μ_s only. These equations state that for a fixed Wiener filter design the mean square distortion, D, due to the use of this filter is independent of variations of μ_n, μ_s, but that the stochastic portion of the mean

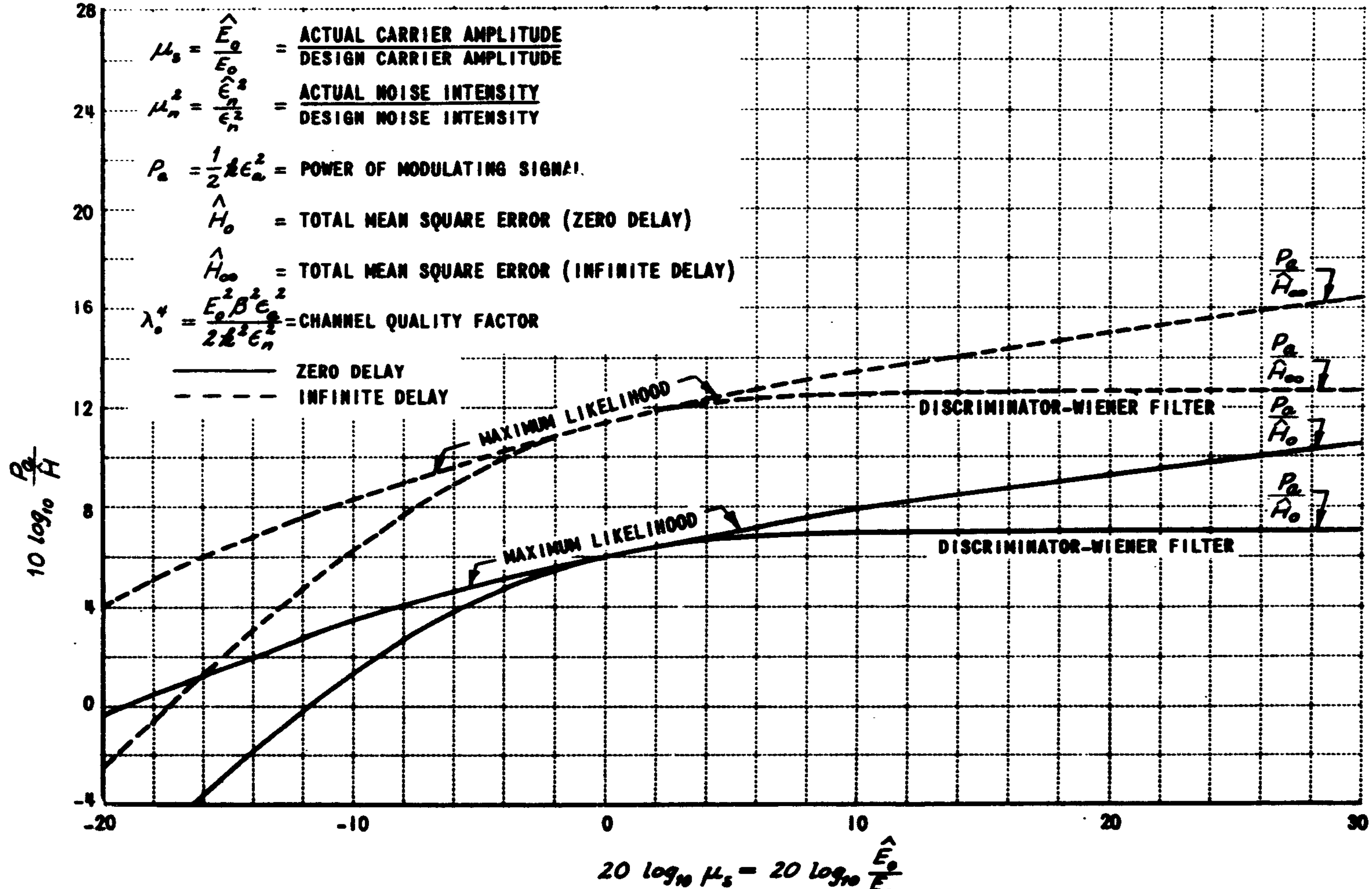


Figure 2 EFFECTS OF VARIATION OF RECEIVED CARRIER STRENGTH ON MEAN SQUARE ERROR

DESIGN CONDITION: $10 \log_{10} \lambda_o^4 = 40$, $\mu_n = 1$

$$\mu_s = \frac{\hat{E}_o}{E_o} = \frac{\text{ACTUAL CARRIER AMPLITUDE}}{\text{DESIGN CARRIER AMPLITUDE}}$$

$$\mu_n^2 = \frac{\hat{\epsilon}_n^2}{\epsilon_n^2} = \frac{\text{ACTUAL NOISE INTENSITY}}{\text{DESIGN NOISE INTENSITY}}$$

$$\frac{\hat{N}}{N} = \frac{\text{MEAN SQUARE ERROR DUE TO NOISE UNDER ACTUAL CONDITIONS}}{\text{MEAN SQUARE ERROR DUE TO NOISE UNDER DESIGN CONDITIONS}}$$

$$\frac{\hat{D}}{D} = \frac{\text{MEAN SQUARE ERROR DUE TO DISTORTION UNDER ACTUAL CONDITIONS}}{\text{MEAN SQUARE ERROR DUE TO DISTORTION UNDER DESIGN CONDITIONS}}$$

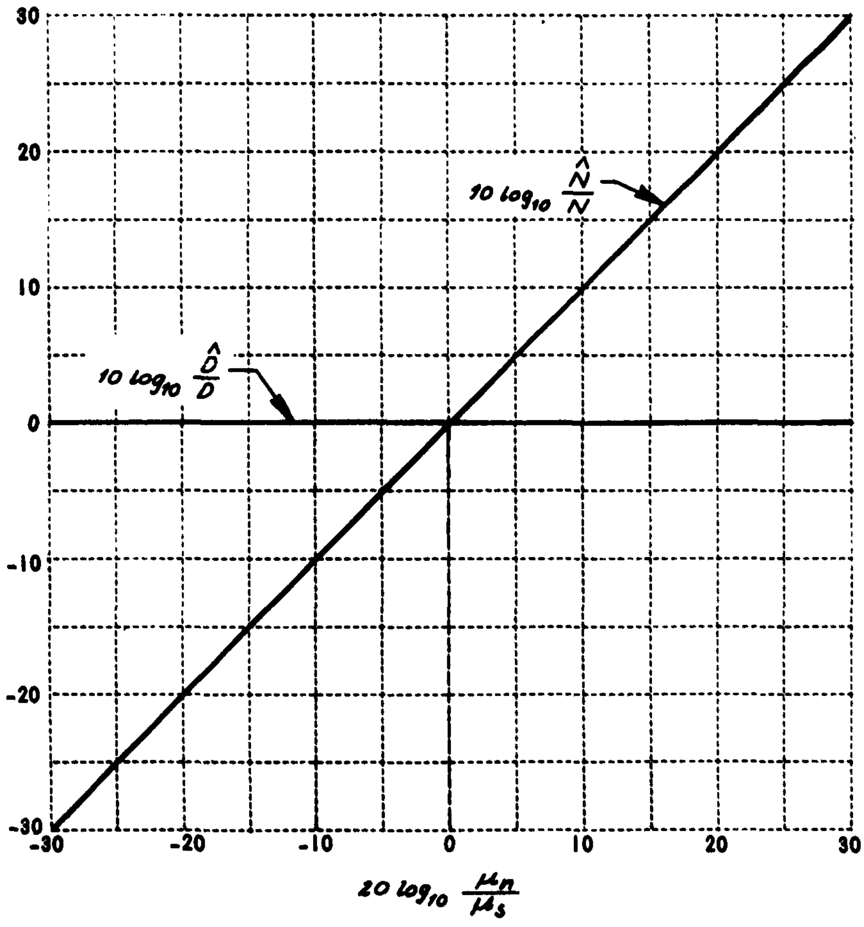


Figure 3 DISCRIMINATOR-WIENER FILTER RECEIVER, 0 OR ∞ DELAY. EFFECT OF OPERATING UNDER OFF-DESIGN CONDITIONS (CURVES ALSO APPLY TO MAXIMUM LIKELIHOOD ESTIMATION FOR $\mu_s = 1$)

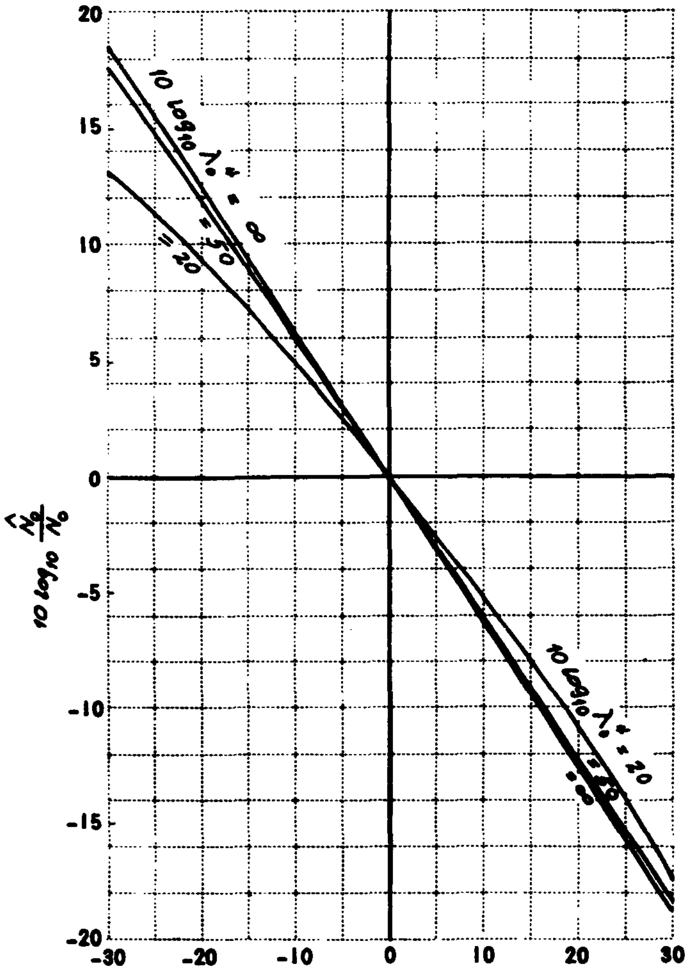


Figure 4a

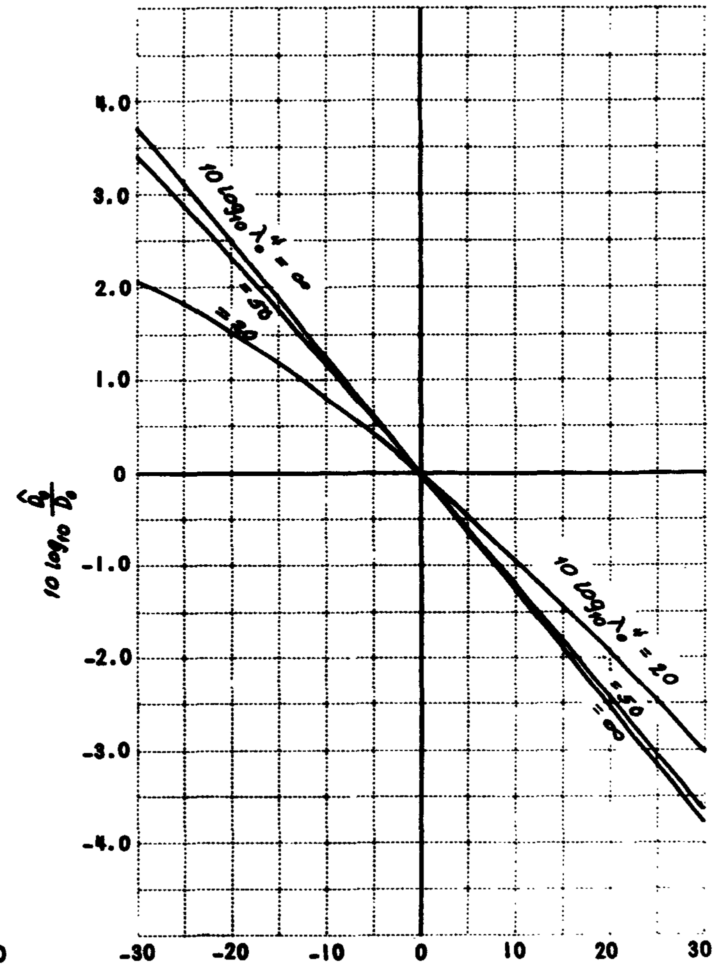


Figure 4b

MAXIMUM LIKELIHOOD ESTIMATION

EFFECTS OF VARIATION OF RECEIVED CARRIER STRENGTH FROM DESIGN CONDITIONS

$\mu_n = 1.0$, ZERO DELAY

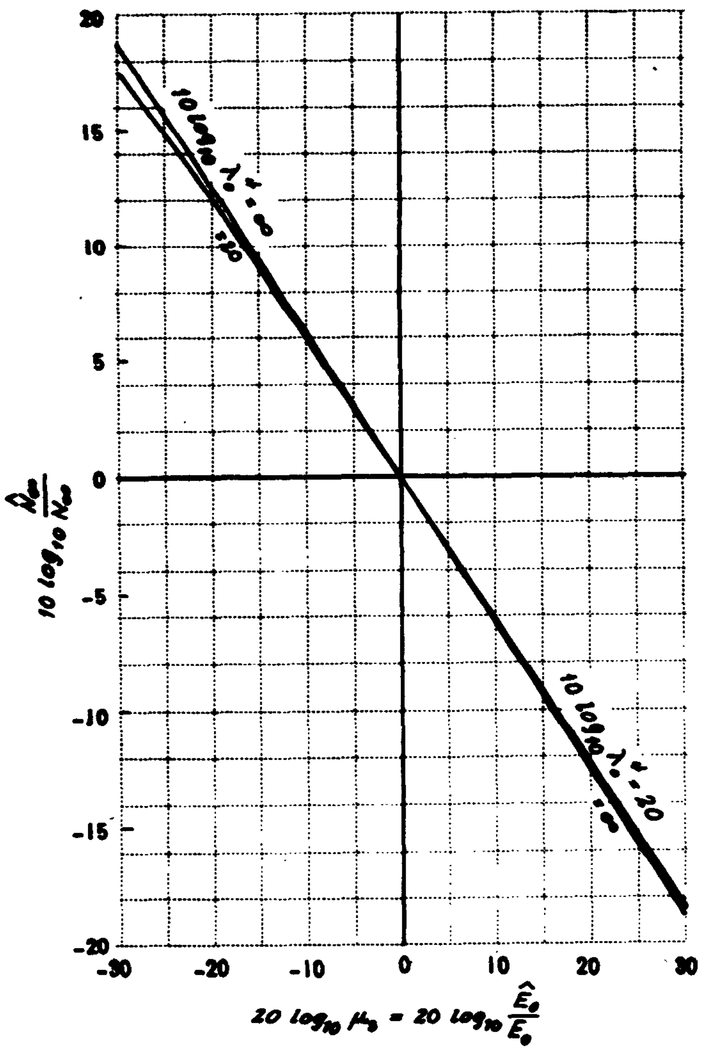


Figure 5a

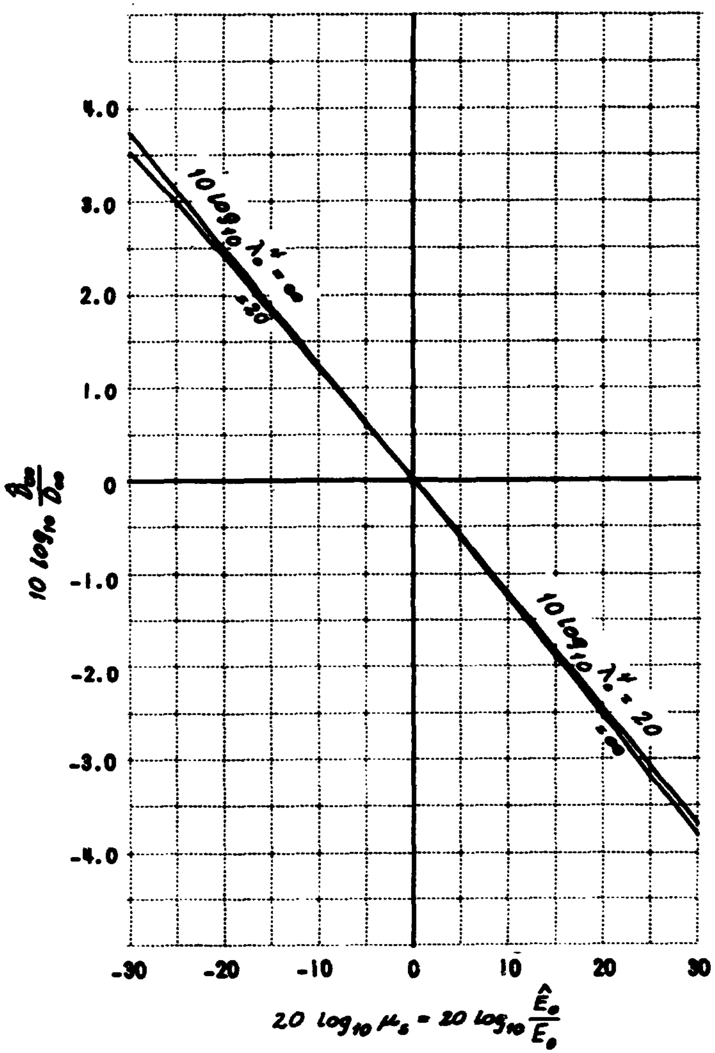


Figure 5b

MAXIMUM LIKELIHOOD ESTIMATION

EFFECTS OF VARIATION OF RECEIVED CARRIER STRENGTH FROM DESIGN CONDITIONS

$\mu_n = 1.0$, INFINITE DELAY

square error, N , is directly proportional to the carrier-to-noise power ratio. From Equations (122), (123), (125) and (126) it will be noted that when $\mu_s = 1$ these plots also describe the performance of maximum likelihood estimation with variation of μ_n .

The effects of variation of received carrier strength on the maximum likelihood estimate are a function of the design point λ_0 . The variation of the total mean square error has already been illustrated for the case $\lambda_0 = 10$ in Figure 2. In Figures 4 and 5 the variation of the components $\hat{N}_0, \hat{D}_0, \hat{N}_\infty, \hat{D}_\infty$ with variation in carrier strength is illustrated. From these curves it is seen that for the range of interest the dependence on λ_0 is not very pronounced. The fact that the plots for $\lambda_0 = \infty$ are straight lines reflects the asymptotic dependence of $\hat{N}$, $\hat{D}$ where $\hat{N} \propto \mu_s^{-5/4}$ and $\hat{D} \propto \mu_s^{1/4}$.

The major significance of our results are: It has been shown, at least for above threshold operation, that the statistically optimum demodulation technique of maximum likelihood estimation yields the same results as obtained by an optimized "inverse" receiver* and that lack of knowledge of the initial carrier phase results in an increased mean square error of the maximum likelihood estimate only during an initial transitory period.

The fact that the assumption of an ideal (Wiener) filter following the ideal discriminator leads to precisely the same results as the maximum likelihood estimation is gratifying and serves as a "check" on a considerable amount of mathematical manipulation. One would, however, not expect great sensitivity to deviations of the filter characteristics from the ideal.

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*By an "inverse" receiver is meant a receiver which performs an operation "inverse" to the modulator, e.g., in FM the modulator produces a rate of change of carrier phase proportional to the modulating signal and the "inverse" receiver produces an output proportional to the rate of change of the receiver phase.

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APPENDIX A -- Exact Solution for $\epsilon(\tau, t)$

Equation (131) may be written

$$\begin{aligned}\epsilon(\tau,t) &= -\frac{E_0\hat{E}_0\beta^2}{2\epsilon_n^2}\cdot\frac{1}{2}k\epsilon_a^2\int_{t-T}^{t}dx\int_{t-T}^{x}dz\,e^{-k|z-\tau|}\int_{t-T}^{x}du\,\epsilon(u,t)\\ &\quad+\frac{E_0\hat{E}_0\beta\Delta\phi}{2\epsilon_n^2}\cdot\frac{1}{2}k\epsilon_a^2\int_{t-T}^{t}dx\int_{t-T}^{x}dz\,e^{-k|z-\tau|}\end{aligned} \quad (A-1)$$

Put

$$\lambda = \left(\frac{E_0\hat{E}_0\beta^2\epsilon_a^2}{2k^2\epsilon_n^2}\right)^{1/4} \quad (A-2)$$

$$\begin{aligned}\chi(\tau,t) &= \frac{E_0\hat{E}_0\beta\Delta\phi}{4\epsilon_n^2}k\epsilon_a^2\int_{t-T}^{t}dx\int_{t-T}^{x}dz\,e^{-k|z-\tau|}\\ &= \frac{1}{2}\lambda^4k^3\beta^{-1}\Delta\phi\int_{t-T}^{t}dx\int_{t-T}^{x}dz\,e^{-k|z-\tau|}\end{aligned} \quad (A-3)$$

then

$$\begin{aligned}\epsilon(\tau,t) &= -\frac{1}{2}k^3\lambda^4\int_{t-T}^{t}dx\int_{t-T}^{x}dz\int_{t-T}^{x}du\,e^{-k|z-\tau|}\epsilon(u,t)+\chi(\tau,t)\\ &= -\frac{1}{2}k^3\lambda^4\int_{t-T}^{t}dz\int_{z}^{t}dx\int_{t-T}^{x}du\,e^{-k|z-\tau|}\epsilon(u,t)+\chi(\tau,t)\end{aligned} \quad (A-4)$$

or

$$\begin{aligned}\epsilon(\tau,t)-\chi(\tau,t) &= -\frac{1}{2}k^3\lambda^4\int_{t-T}^{\tau}dz\int_{z}^{t}dx\int_{t-T}^{x}du\,e^{-k(\tau-z)}\epsilon(u,t)\\ &\quad-\frac{1}{2}k^3\lambda^4\int_{\tau}^{t}dz\int_{z}^{t}dx\int_{t-T}^{x}du\,e^{k(\tau-z)}\epsilon(u,t)\end{aligned} \quad (A-5)$$

Letting $\quad D \equiv \frac{\partial}{\partial \tau}$,

$$D(\epsilon-\chi) = \frac{1}{2} k^4 \lambda^4 \int_{t-T}^{\tau} dz \int_{z}^{t} dx \int_{t-T}^{x} du\, e^{-k(\tau-z)} \epsilon(u,t) - \frac{1}{2} k^4 \lambda^4 \int_{\tau}^{t} dz \int_{z}^{t} dx \int_{t-T}^{x} du\, e^{k(\tau-z)} \epsilon(u,t) \tag{A-6}$$

$$\begin{aligned} D^2(\epsilon-\chi) &= -\frac{1}{2} k^5 \lambda^4 \int_{t-T}^{\tau} dz \int_{z}^{t} dx \int_{t-T}^{x} du\, e^{-k(\tau-z)} \epsilon(u,t) \\ &\quad + \frac{1}{2} k^4 \lambda^4 \int_{\tau}^{t} dx \int_{t-T}^{x} du\, \epsilon(u,t) \\ &\quad - \frac{1}{2} k^5 \lambda^4 \int_{\tau}^{t} dz \int_{z}^{t} dx \int_{t-T}^{x} du\, e^{k(\tau-z)} \epsilon(u,t) \\ &\quad + \frac{1}{2} k^4 \lambda^4 \int_{\tau}^{t} dx \int_{t-T}^{x} du\, \epsilon(u,t) \\ &= k^2(\epsilon-\chi) + k^4 \lambda^4 \int_{\tau}^{t} dx \int_{t-T}^{x} du\, \epsilon(u,t) \end{aligned} \tag{A-7}$$

$$(D^2 - k^2)(\epsilon-\chi) = k^4 \lambda^4 \int_{\tau}^{t} dx \int_{t-T}^{x} du\, \epsilon(u,t) \tag{A-8}$$

$$(D^3 - k^2 D)(\epsilon-\chi) = -k^4 \lambda^4 \int_{t-T}^{\tau} du\, \epsilon(u,t) \tag{A-9}$$

$$(D^4 - k^2 D^2)(\epsilon-\chi) = -k^4 \lambda^4 \epsilon(\tau,t) \tag{A-10}$$

$$(D^4 - k^2 D^2 + k^4 \lambda^4)\epsilon = (D^4 - k^2 D^2)\chi \tag{A-11}$$

Now $$(\epsilon - \chi)_{\tau = t} = -\frac{1}{2} k^3 \lambda^4 \int_{t-T}^{t} dz \int_{z}^{t} dx \int_{t-T}^{x} du\, e^{-k(t-z)} \epsilon(u,t) \tag{A-12}$$

$$(\epsilon - \chi)_{\tau = t-T} = -\frac{1}{2} k^3 \lambda^4 \int_{t-T}^{t} dz \int_{\tau}^{t} dx \int_{t-T}^{x} du\, e^{-k(z-t+T)} \epsilon(u,t) \tag{A-13}$$

$$D(\epsilon - \chi)_{\tau = t} = \frac{1}{2} k^4 \lambda^4 \int_{t-T}^{t} dz \int_{z}^{t} dx \int_{t-T}^{x} du\, e^{-k(t-z)} \epsilon(u,t) \tag{A-14}$$

$$D(\epsilon - \chi)_{\tau = t-T} = -\frac{1}{2} k^4 \lambda^4 \int_{t-T}^{t} dz \int_{z}^{t} dx \int_{t-T}^{x} du\, e^{-k(z-t+T)} \epsilon(u,t) \tag{A-15}$$

Therefore,

$$(D+k)(\epsilon - \chi)_{\tau = t} = 0, \quad (D-k)(\epsilon - \chi)_{\tau = t-T} = 0 \tag{A-16}$$

It is clear that

$$(D^2 - k^2)(\epsilon - \chi)_{\tau = t} = 0, \quad (D^3 - k^2 D)(\epsilon - \chi)_{\tau = t-T} = 0 \tag{A-17}$$

Now from A-3

$$\begin{aligned} \chi(\tau, t) &= \frac{1}{2} \lambda^4 k^3 \beta^{-1} \Delta\phi \int_{t-T}^{t} dz \int_{z}^{t} dx\, e^{-k|z-\tau|} \\ &= \frac{1}{2} \lambda^4 k^3 \beta^{-1} \Delta\phi \int_{t-T}^{\tau} dz\, (t-z)\, e^{-k(\tau-z)} \\ &\quad + \frac{1}{2} \lambda^4 k^3 \beta^{-1} \Delta\phi \int_{\tau}^{t} dz\, (t-z)\, e^{+k(\tau-z)} \end{aligned} \tag{A-18}$$

$$\begin{aligned} D\chi &= -\frac{1}{2} \lambda^4 k^4 \beta^{-1} \Delta\phi \int_{t-T}^{\tau} dz\, (t-z)\, e^{-k(\tau-z)} \\ &\quad + \frac{1}{2} \lambda^4 k^4 \beta^{-1} \Delta\phi \int_{\tau}^{t} dz\, (t-z)\, e^{+k(\tau-z)} \end{aligned} \tag{A-19}$$

$$\begin{aligned} D^2\chi &= \frac{1}{2} \lambda^4 k^5 \beta^{-1} \Delta\phi \int_{t-T}^{\tau} dz\, (t-z)\, e^{-k(\tau-z)} - \frac{1}{2} \lambda^4 k^4 \beta^{-1} \Delta\phi (t-\tau) \\ &\quad + \frac{1}{2} \lambda^4 k^5 \beta^{-1} \Delta\phi \int_{\tau}^{t} dz\, (t-z)\, e^{k(\tau-z)} - \frac{1}{2} \lambda^4 k^4 \beta^{-1} \Delta\phi (t-\tau) \end{aligned} \tag{A-20}$$

$$(D^2 - k^2)\chi = -\lambda^4 k^4 \beta^{-1} \Delta\phi (t - \tau) \tag{A-21}$$

$$(D^3 - k^2 D)\chi = \lambda^4 k^4 \beta^{-1} \Delta\phi \tag{A-22}$$

$$(D^4 - k^2 D^2)\chi = 0 \tag{A-23}$$

$$\begin{aligned} \chi(t,t) &= \frac{1}{2}\lambda^4 k^3 \beta^{-1} \Delta\phi \int_{t-T}^{t} dz\,(t-z)\,e^{-k(t-z)}; \\ \chi(t-T,t) &= \frac{1}{2}\lambda^4 k^3 \beta^{-1} \Delta\phi \int_{t-T}^{t} dz\,(t-z)\,e^{-k(z-t+T)} \end{aligned} \tag{A-24}$$

$$\begin{aligned} D\chi_{\tau=t} &= -\frac{1}{2}\lambda^4 k^4 \beta^{-1} \Delta\phi \int_{t-T}^{t} dz\,(t-z)\,e^{-k(t-z)}; \\ D\chi_{\tau=t-T} &= \frac{1}{2}\lambda^4 k^4 \beta^{-1} \Delta\phi \int_{t-T}^{t} dz\,(t-z)\,e^{-k(z-t+T)} \end{aligned} \tag{A-25}$$

So $\quad (D+k)\chi_{\tau=t} = 0 \qquad (D-k)\chi_{\tau=t-T} = 0 \qquad$ (A-26)

Therefore, we have for ϵ the differential equation

$$(D^4 - k^2 D^2 + k^4 \lambda^4)\epsilon = 0 \tag{A-27}$$

and the following four boundary conditions.

$$\Big[(D+k)\epsilon\Big]_{\tau=t} = 0 \tag{A-28}$$

$$\Big[(D-k)\epsilon\Big]_{\tau=t-T} = 0 \tag{A-29}$$

$$\Big[(D^2-k^2)\epsilon\Big]_{\tau=t} = 0 \tag{A-30}$$

$$\Big[(D^3-k^2 D)\epsilon\Big]_{\tau=t-T} = k^4 \lambda^4 \frac{\Delta\phi}{\beta} \tag{A-31}$$

The general solution of A-27 may be written

$$\epsilon(\tau,t) = e^{\bar{c}\eta}(a_1 \cos \hat{c}y + a_2 \sin \hat{c}y) + e^{-\bar{c}y}(a_3 \cos \hat{c}y + a_4 \sin \hat{c}y) \qquad \text{A-32}$$

where $\eta = \tau - t - T;\ y = \tau - t + T_0$. $c = \bar{c} + i\hat{c}$, $c^* = \bar{c} - i\hat{c}$, $-c$, and $-c^*$ are the four roots of $D^4 - k^2 D^2 + k^4\lambda^4 = 0$ so that*

$$\left.\begin{aligned} \bar{c} &= \frac{k}{2}\sqrt{2\lambda^2+1} \\ \hat{c} &= \frac{k}{2}\sqrt{2\lambda^2-1} \end{aligned}\right\} \qquad \text{A-33}$$

The coefficients a_i are functions independent of τ and, for the required solution, are to be determined by conditions A-28 to A-31.

From A-32 one obtains:

$$\begin{aligned} D\epsilon &= e^{\bar{c}\eta}\{(\bar{c}a_1 + \hat{c}a_2)\cos\hat{c}y + (\bar{c}a_2 - \hat{c}a_1)\sin\hat{c}y\} \\ &\quad - e^{-\bar{c}y}\{(\bar{c}a_3 - \hat{c}a_4)\cos\hat{c}y + (\bar{c}a_4 + \hat{c}a_3)\sin\hat{c}y\} \end{aligned}$$

$$\begin{aligned} D^2\epsilon &= \frac{k^2}{2}e^{\bar{c}\eta}\{(a_1 + \sqrt{4\lambda^4-1}\,a_2)\cos\hat{c}y + (a_2 - \sqrt{4\lambda^4-1}\,a_1)\sin\hat{c}y\} \\ &\quad + \frac{k^2}{2}e^{-\bar{c}y}\{(a_3 - \sqrt{4\lambda^4-1}\,a_4)\cos\hat{c}y + (a_4 + \sqrt{4\lambda^4-1}\,a_3)\sin\hat{c}y\} \end{aligned}$$

$$\begin{aligned} D^3\epsilon = \frac{k^2}{2}e^{\bar{c}\eta}\Big[&\{\bar{c}(a_1 + \sqrt{4\lambda^4-1}\,a_2) + \hat{c}(a_2 - \sqrt{4\lambda^4-1}\,a_1)\}\cos\hat{c}y \\ &+ \{\bar{c}(a_2 - \sqrt{4\lambda^4-1}\,a_1) - \hat{c}(a_1 + \sqrt{4\lambda^4-1}\,a_2)\}\sin\hat{c}y\Big] \\ - \frac{k^2}{2}e^{-\bar{c}y}\Big[&\{\bar{c}(a_3 - \sqrt{4\lambda^4-1}\,a_4) - \hat{c}(a_4 + \sqrt{4\lambda^4-1}\,a_3)\}\cos\hat{c}y \\ &+ \{\bar{c}(a_4 + \sqrt{4\lambda^4-1}\,a_3) + \hat{c}(a_3 - \sqrt{4\lambda^4-1}\,a_4)\}\sin\hat{c}y\Big] \end{aligned}$$

---

*We are only interested in the cases where $\lambda^2 > 1/2$.

Using these relations together with A-32, condition A-28 becomes

$$c_1 a_1 + c_2 a_2 + c_3 a_3 + c_4 a_4 = 0 \qquad \text{(A-34)}$$

where

$$\left.\begin{aligned}
c_1 &= \frac{\sqrt{2\lambda^2+1}+2}{\sqrt{2\lambda^2-1}} \cos \hat{c}T - \sin \hat{c}T \\
c_2 &= \cos \hat{c}T + \frac{\sqrt{2\lambda^2+1}+2}{\sqrt{2\lambda^2-1}} \sin \hat{c}T \\
c_3 &= -\left\{ \frac{\sqrt{2\lambda^2+1}-2}{\sqrt{2\lambda^2-1}} \cos \hat{c}T + \sin \hat{c}T \right\} \\
c_4 &= \cos \hat{c}T - \frac{\sqrt{2\lambda^2+1}-2}{\sqrt{2\lambda^2-1}} \sin \hat{c}T
\end{aligned}\right\} \qquad \text{(A-35)}$$

Similarly, condition A-29 becomes

$$(\bar{c}-k)e^{-2\bar{c}T}a_1 + \hat{c}e^{-2\bar{c}T}a_2 - (\bar{c}+k)a_3 + \hat{c}a_4 = 0 \qquad \text{(A-36)}$$

condition A-30 becomes

$$b_1 a_1 + b_2 a_2 + b_3 a_3 + b_4 a_4 = 0 \qquad \text{(A-37)}$$

where

$$\left.\begin{aligned}
b_1 &= \cos \hat{c}T + \sqrt{4\lambda^4-1} \sin \hat{c}T \\
b_2 &= -\sqrt{4\lambda^4-1} \cos \hat{c}T + \sin \hat{c}T \\
b_3 &= \cos \hat{c}T - \sqrt{4\lambda^4-1} \sin \hat{c}T \\
b_4 &= \sqrt{4\lambda^4-1} \cos \hat{c}T + \sin \hat{c}T
\end{aligned}\right\} \qquad \text{(A-37a)}$$

and using

$$\bar{c}\sqrt{4\lambda^4-1} - \hat{c} = 2\lambda^2\hat{c} \qquad \text{and} \qquad \bar{c} + \hat{c}\sqrt{4\lambda^4-1} = 2\lambda^2\bar{c}$$

condition A-31 becomes

$$-\bar{c}e^{-2\bar{c}T}a_1 + \hat{c}e^{-2\bar{c}T}a_2 + \bar{c}a_3 + \hat{c}a_4 = k^2\lambda^2 \frac{\Delta\phi}{\beta} \qquad \text{(A-38)}$$

From A-36 and A-38, one obtains

$$a_3 = \frac{1}{u}\left\{k\lambda^2\frac{\Delta\phi}{\beta} + \left(\sqrt{2\lambda^2+1}-1\right)e^{-2\bar{\sigma}T}a_1\right\} \tag{A-39}$$

and
$$a_4 = \frac{1}{u}\left\{\frac{2+\sqrt{2\lambda^2+1}}{\sqrt{2\lambda^2-1}}k\lambda^2\frac{\Delta\phi}{\beta} + 2\sqrt{\frac{2\lambda^2+1}{2\lambda^2-1}}e^{-2\bar{\sigma}T}a_1\right\} - e^{-2\bar{\sigma}T}a_2 \tag{A-40}$$

where
$$u = \frac{2\bar{c}+k}{k} = 1+\sqrt{2\lambda^2+1} \tag{A-41}$$

Substituting A-38 and A-40 in A-35 and A-37 results in

$$Aa_1 + u(c_2 - c_4 e^{-2\bar{\sigma}T})a_2 = -\left(c_3 + \frac{2+\sqrt{2\lambda^2+1}}{\sqrt{2\lambda^2-1}}c_4\right)k\lambda^2\frac{\Delta\phi}{\beta} \tag{A-42}$$

and
$$Ba_1 + u(b_2 - b_4 e^{-2\bar{\sigma}T})a_2 = -\left(b_3 + \frac{2+\sqrt{2\lambda^2+1}}{\sqrt{2\lambda^2-1}}b_4\right)k\lambda^2\frac{\Delta\phi}{\beta} \tag{A-43}$$

where
$$\left.\begin{aligned} A &= uc_1 + \left\{\left(\sqrt{2\lambda^2+1}-1\right)c_3 + 2\sqrt{\frac{2\lambda^2+1}{2\lambda^2-1}}c_4\right\}e^{-2\bar{\sigma}T} \\ B &= ub_1 + \left\{\left(\sqrt{2\lambda^2+1}-1\right)b_3 + 2\sqrt{\frac{2\lambda^2+1}{2\lambda^2-1}}b_4\right\}e^{-2\bar{\sigma}T} \end{aligned}\right\} \tag{A-44}$$

Solving for a_1 and a_2 from A-42 and A-43, one obtains

$$\begin{aligned} a_1 = \frac{1}{V}k\lambda^2\frac{\Delta\phi}{\beta}\Bigg\{&(b_2 - b_4 e^{-2\bar{\sigma}T})\left(c_3 + \frac{2+\sqrt{2\lambda^2+1}}{\sqrt{2\lambda^2-1}}c_4\right) \\ &-(c_2 - c_4 e^{-2\bar{\sigma}T})\left(b_3 + \frac{2+\sqrt{2\lambda^2+1}}{\sqrt{2\lambda^2-1}}b_4\right)\Bigg\} \end{aligned} \tag{A-45}$$

$$\begin{aligned} a_2 = -\frac{1}{V}k\lambda^2\frac{\Delta\phi}{\beta}\Bigg\{&\frac{A}{u}\left(b_3 + \frac{2+\sqrt{2\lambda^2+1}}{\sqrt{2\lambda^2-1}}b_4\right) \\ &-\frac{B}{u}\left(c_3 + \frac{2+\sqrt{2\lambda^2+1}}{\sqrt{2\lambda^2-1}}c_4\right)\Bigg\} \end{aligned} \tag{A-46}$$

where

$$V = (c_2 - c_4 e^{-2\bar{c}T})B - (b_2 - b_4 e^{-2\bar{c}T})A \tag{A-47}$$

After a_1 and a_2 are calculated from A-45 and A-46, a_3 and a_4 can be calculated from A-39 and A-40.

When $\tau = t$, A-32 becomes

$$\epsilon(t,t) = e^{-\bar{c}T}\{(a_1 + a_3)\cos\hat{c}T + (a_2 + a_4)\sin\hat{c}T\}$$

If
$$T \gg \frac{1}{k\sqrt{2\lambda^2+1}}$$

then
$$e^{-2\bar{c}T} = e^{-k\sqrt{2\lambda^2+1}\,T} \approx 0$$

and we have:

$$A \approx u c_1$$

$$B \approx u b_1$$

$$V \approx u(b_1 c_2 - b_2 c_1)$$

$$= 2u(1 + \lambda^2 + \sqrt{2\lambda^2+1}) \quad \text{(By A-35 and A-37a)}$$

$$> 7 + 5\sqrt{2} \qquad \text{(Since we assumed } \lambda^2 > \tfrac{1}{2}\text{)}$$

and, hence, all the coefficients a_i are finite so that $\epsilon(t,t) \to 0$ as $T \to \infty$.

In order to show that $\epsilon(\tau,t) \to 0$ for all $\tau - (t - T) \equiv y \gg \frac{1}{k\sqrt{2\lambda^2+1}}$

we rewrite A-32 as

$$\epsilon(\tau,t) = e^{\bar{c}(y-2T)}(a_1 \cos\hat{c}y + a_2 \sin\hat{c}y)$$
$$+ e^{-\bar{c}y}(a_3 \cos\hat{c}y + a_4 \sin\hat{c}y), \qquad 0 \le y \le T$$

$$|\epsilon(\tau,t)| \le e^{-\bar{c}y}[|a_1| + |a_2| + |a_3| + |a_4|] = O(e^{-\bar{c}y})$$

APPENDIX B -- Evaluation of Certain Integrals

$$I_1 = \int_{-\infty}^{\infty} \frac{\omega^2}{\omega^4+K^2\omega^2+K^4\lambda_0^4}\, d\omega$$

$$I_2 = \int_{-\infty}^{\infty} \frac{1}{\omega^4+K^2\omega^2+K^4\lambda_0^4}\, d\omega$$

$$I_3 = \int_{-\infty}^{\infty} \frac{\omega^2}{(\omega^2+K^2)(\omega^4+K^2\omega^2+K^4\lambda_0^4)}\, d\omega$$

$$I_4 = \int_{-\infty}^{\infty} \frac{1}{(\omega^2+K^2)(\omega^4+K^2\omega^2+K^4\lambda_0^4)}\, d\omega$$

$$I_5 = \int_{-\infty}^{\infty} \frac{\omega^2}{(\omega^4+K^2\omega^2+K^4\lambda_0^4)^2}\, d\omega$$

By writing

$$\begin{aligned}
&\omega^4+K^2\omega^2+K^4\lambda_0^4 \\
&\quad = \left\{\omega^2+(b_2^2+b_1^2)-2b_2\omega\right\}\cdot\left\{\omega^2+(b_2^2+b_1^2)+2b_2\omega\right\} \\
&\quad = \left\{\omega-(b_2+ib_1)\right\}\cdot\left\{\omega-(b_2-ib_1\right\} \\
&\qquad\qquad \cdot\left\{\omega-(-b_2+ib_1)\right\}\cdot\left\{\omega-(-b_2-ib_1)\right\}
\end{aligned}$$

where

$$b_1 = \frac{K}{2}\sqrt{2\lambda_0^2+1}$$

$$b_2 = \frac{K}{2}\sqrt{2\lambda_0^2-1}$$

and then using the method of residues, one finds:

$$I_1 = 2\pi i \left\{ \frac{(b_2+ib_1)^2}{(i2b_1)(2b_2)\,2(b_2+ib_1)} + \frac{(-b_2+ib_1)^2}{(-2b_2)\,2(-b_2+ib_1)(i2b_1)} \right\}$$

$$= \frac{\pi}{4b_2b_1}\left\{(b_2+ib_1)-(-b_2+ib_1)\right\} = \frac{\pi}{K}\frac{1}{\sqrt{2\lambda_0^2+1}}$$

$$I_2 = \frac{\pi}{4b_2b_1}\left(\frac{1}{b_2+ib_1} - \frac{1}{-b_2+ib_1}\right) = \frac{\pi}{K^3}\frac{1}{\lambda_0^2\sqrt{2\lambda_0^2+1}}$$

$$I_3 = 2\pi i\left[\frac{-K^2}{i2K(K^4\lambda_0^4)} + \frac{1}{i8b_2b_1}\left\{\frac{b_2+ib_1}{(b_2+ib_1)^2+K^2} - \frac{-b_2+ib_1}{(-b_2+ib_1)^2+K^2}\right\}\right]$$

$$= \pi\left[-\frac{1}{K^3\lambda_0^4} + \frac{1}{4b_2b_1}\left\{\frac{b_2+ib_1}{\frac{K^2}{2}+i2b_2b_1} - \frac{-b_2+ib_1}{\frac{K^2}{2}-i2b_2b_1}\right\}\right]$$

$$= \pi\left\{-\frac{1}{K^3\lambda_0^4} + \frac{1}{2b_1}\;\frac{1}{K^2\lambda_0^4}(\lambda_0^2+1)\right\}$$

$$= \frac{\pi}{K^3}\frac{1}{\lambda_0^4}\left(\frac{\lambda_0^2+1}{\sqrt{2\lambda_0^2+1}} - 1\right)$$

$$I_4 = \pi\left[\frac{+1}{K^5\cdot\lambda_0^4} + \frac{1}{4b_2b_1}\left\{\frac{(b_2+ib_1)^{-1}}{\frac{K^2}{2}+i2b_2b_1} - \frac{(-b_2+ib_1)^{-1}}{\frac{K^2}{2}-i2b_2b_1}\right\}\right]$$

$$= \pi\left(\frac{1}{K^5\lambda_0^4} - \frac{1}{2b_1}\;\frac{1}{K^4\lambda_0^4}\right) = \frac{\pi}{K^5}\frac{1}{\lambda_0^4}\left(1-\frac{1}{\sqrt{2\lambda_0^2+1}}\right)$$

$$I_5 = 2\pi i\left[\frac{2(b_2+ib_1)}{(i2b_1)^2(2b_2)^2\,4(b_2+ib_1)^2}\left\{1-\frac{b_2+ib_1}{i2b_1}-\frac{b_2+ib_1}{2b_2}-\frac{b_2+ib_1}{2(b_2+ib_1)}\right\}\right.$$

$$\left.+\frac{2(-b_2+ib_1)}{(-2b_2)^2\,4(-b_2+ib_1)^2(i2b_1)^2}\left\{1-\frac{-b_2+ib_1}{-2b_2}-\frac{-b_2+ib_1}{2(-b_2+ib_1)}-\frac{-b_2+ib_1}{i2b_1}\right\}\right]$$

$$= \frac{2\pi i}{-64b_1^2b_2^2}\left[\left\{\frac{1}{b_2+ib_1}-\frac{1}{ib_1}-\frac{1}{b_2}\right\}+\left\{\frac{1}{-b_2+ib_1}+\frac{1}{b_2}-\frac{1}{ib_1}\right\}\right]$$

$$= \frac{\pi}{16b_1^2b_2^2}\left(\frac{1}{b_1}-\frac{b_1}{b_2^2+b_1^2}\right) = \frac{\pi}{16b_1^3(b_2^2+b_1^2)}$$

$$= \frac{\pi}{2K^3(\sqrt{2\lambda_0^2+1})^3(K^2\lambda_0^2)}$$

$$= \frac{\pi}{K^5}\frac{1}{2\lambda_0^2(\sqrt{2\lambda_0^2+1})^3}$$

In the computation of I_5 , the residues at the second-order poles were evaluated by use of the relations

$$\text{Res. at } a_1 = \frac{d}{d\omega}\left\{(\omega-a_1)^2 f(\omega)\right\}_{\omega=a_1}$$

and

$$\frac{d}{d\omega}\left\{\frac{\omega^2}{(\omega-\chi_1)^2(\omega-\chi_2)^2(\omega-\chi_3)^2}\right\}$$

$$= \frac{2\omega}{(\omega-\chi_1)^2(\omega-\chi_2)^2(\omega-\chi_3)^2}\left\{1-\frac{\omega}{\omega-\chi_1}-\frac{\omega}{\omega-\chi_2}-\frac{\omega}{\omega-\chi_3}\right\}$$

Defining

$$\gamma_0 \equiv \sqrt{2\lambda_0^2+1}$$

the above results take the form:

$$I_1 = \frac{\pi}{K}\frac{1}{\gamma_0}$$

$$I_2 = \frac{\pi}{K^3}\frac{2}{\gamma_0(\gamma_0^2-1)}$$

$$I_3 = \frac{\pi}{K^3}\frac{2}{\gamma_0(\gamma_0+1)^2}$$

$$I_4 = \frac{\pi}{K^5}\frac{4}{\gamma_0(\gamma_0-1)(\gamma_0+1)^2}$$

$$I_5 = \frac{\pi}{K^5}\frac{1}{\gamma_0^3(\gamma_0^2-1)}$$

III

THRESHOLD EFFECTS IN FM RECEIVERS WITH RANDOMLY MODULATED SIGNALS

SUMMARY

The threshold phenomenon in an FM receiver which consists of an ideal discriminator and a post-detection Wiener filter is examined for the case when the modulating signal is a gaussian random process with zero mean. For this type of modulating signal, the power spectrum of the discriminator output noise can be obtained by an approach due to Rice[1]. Three difference cases are treated: (1) the power spectrum of the modulating signal is similar to that of a white noise passed through a first-order low pass filter, and an infinite delay Wiener filter is used; (2) the signal spectrum is as in Case (1), but a zero delay Wiener filter is used; (3) the signal spectrum is constant in a limited band and zero outside, and an infinite delay Wiener filter is used. It is found that the carrier-to-noise (in the I. F. bandwidth) ratio at which threshold occurs depends on the modulation and I. F. bandwidth. Graphs showing performance near threshold are presented.

1. Power Spectrum of Discriminator Output Noise

The signal transmitted to the FM receiver has the form

$$E_0 \cos\left\{ \omega_c t + \beta \int_0^t x(\tau)\, d\tau + \alpha \right\}$$

in which ω_c , β , and α are constants. The discriminator gain can be determined as β^{-1} by requiring that its output reproduce the modulating signal $x(\tau)$ in the absence of noise.

Rice conjectures, Equation (2.31) of Reference 1, that the two-sided power spectrum, $N_D(\omega)$, of the output noise of a discriminator with gain β^{-1} is given by*

$$N_D(\omega) \approx \frac{1}{\beta^2} \left\{ 4\pi^2 (N_+ + N_-) + \frac{\omega^2}{E_0^2} W_y(\omega) \right\} \tag{1}$$

*Note that one-sided spectra were used in Reference 1, and two-sided spectra are used here.

where E_0 is the carrier amplitude and N_+ and N_- are the expected number of times per second that the discriminator input noise phase increases and decreases by an odd multiple of π radians, respectively. $W_y(\omega)$ is the two-sided power spectrum of the input noise component in quadrature with the modulated carrier,

$$y(t) = n_s(t)\cos\varphi(t) - n_c(t)\sin\varphi(t) \tag{2}$$

where $\varphi(t)$ is the carrier phase at time t resulting from the modulating signal, and $n_c(t)$ and $n_s(t)$ are, respectively, the in-phase and quadrature components of the noise with respect to the unmodulated carrier.

We are concerned with the case where the input noise to the receiver and the modulating signal are both gaussian with zero mean. In this case, $N_+ = N_-$. The noise is also assumed to be white with power spectral density $\mathcal{E}_n^2$.

We are going to consider two gaussian random processes having different power spectra, but the same total power, as generating the modulating signal. Let the power spectra be

$$S_a(\omega) = \mathcal{E}_a^2 \frac{K^2}{\omega^2 + K} \qquad (-\infty < \omega < \infty) \tag{3}$$

$$S_b(\omega) = \begin{cases} \mathcal{E}_b^2 = \dfrac{\mathcal{E}_a^2}{1-v}, & \Omega_1 \leq |\omega| \leq \Omega_2 \\ 0, & \text{ELSEWHERE} \end{cases} \tag{4}$$

where

$$\Omega_1 = v\,\Omega_2 \qquad (0 \leq v < 1)$$

$$\Omega_2 = \frac{\pi}{2} K$$

so that the total power

$$P = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_a(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_b(\omega)\, d\omega = \frac{1}{2}\,\mathcal{E}_a^2\, K \tag{5}$$

(The spectra have been normalized in such a manner that the total powers are the same so that the results obtained will be applicable to all cases.)

Let B_{IF} be the bandwidth of the I. F. filter which is assumed to have a rectangular passband and to be phase compensated. The power spectral density of $n_s(t)$ or $n_c(t)$ at the discriminator input is then

$$\begin{cases} 2\,\mathcal{E}_n^2 & \text{FOR } |\omega| \leq \pi B_{IF} \\ 0 & \text{FOR } |\omega| > \pi B_{IF} \end{cases} \tag{6}$$

and the carrier-to-noise power ratio, ρ, at the discriminator input is given by

$$\rho = \frac{E_0^2}{4\,\mathcal{E}_n^2\, B_{IF}} \tag{7}$$

In order to avoid distortion of the signal, the I. F. bandwidth must be wide enough to cover all essential spectral components of the modulated carrier. On the other hand, in order to keep the carrier-to-noise ratio at the input to the discriminator large, it is desirable to restrict the I. F. bandwidth as much as possible. If the signals were limited to an amplitude $\pm A$ and maximum frequency $\frac{\Omega_2}{2\pi} = \frac{K}{4}$, then B_{IF} would be given approximately by

$$B_{IF} = \frac{\beta}{2\pi}\, 2A + \frac{K}{2} \tag{8}$$

Although the spectrum $S_a(\omega)$ has no well-defined maximum frequency, its noise equivalent bandwidth is $K/2$ cps (two-sided spectrum). Since the modulating signals, $a(\tau)$ and $b(\tau)$, are assumed to be gaussianly distributed, a maximum amplitude A cannot be rigorously specified*.

*Distributions encountered in practice differ from true gaussian distributions in that their tails do not extend to $\pm\infty$.

However, the probability that $|a(\tau)| > A$ or $|b(\tau)| > A$ is given by

$$P\{|a(\tau)| > A\} = 1 - erf\left(\frac{A}{\sqrt{2P}}\right)$$

which is less than 10^{-7} if we choose $A = 4\sqrt{2P}$. The required B_{IF} for this choice of A is, by Equation (8),

$$B_{IF} = \frac{4}{\pi} \cdot \beta \sqrt{2P} + \frac{k}{2} \tag{9}$$

It is believed that when B_{IF} is specified by Equation (9), the distortion of the output signals due to the I. F. filter is negligible. Since one may wish to choose a value of B_{IF} different from (narrower than) that specified by Equation (9), in the analysis we will use

$$B_{IF} = \frac{m}{\pi} \beta \sqrt{2P} + \frac{k}{2} \tag{10}$$

with m unspecified. Defining θ, a nondimensional modulation parameter, by

$$\theta \equiv \frac{\beta}{k} \sqrt{2P} = \frac{\beta}{k} \sqrt{\mathcal{E}_a^2 k}\,. \tag{11}$$

Then Equation (10) may be written as

$$B_{IF} = \frac{k}{\pi}\left(m\theta + \frac{\pi}{2}\right) \tag{12}$$

For the case of a gaussianly distributed modulating signal, according to Equation (5. 13) of Reference 1, ,

$$N_+ = \frac{1}{\sqrt{\pi}} \frac{B_{IF}}{\sqrt{12}} \int_{\sqrt{p}}^{\infty} e^{-u^2} \sqrt{1 + 2au^2}\, du \tag{13}$$

where

$$a = \beta^2 \frac{\rho}{(2\pi B_{IF}/\sqrt{12})^2} = \frac{3}{2}\left(\frac{k}{\pi B_{IF}}\right)^2 \theta^2 \tag{14}$$

Since

$$\int_{\sqrt{\rho}}^{\infty} e^{-u^2}\sqrt{1+2au^2}\,du$$

$$= \int_{u=\sqrt{\rho}}^{u=\infty} \frac{\sqrt{1+2au^2}}{u}\, d\left(-\frac{1}{2}e^{-u^2}\right)$$

$$= e^{-\rho}\sqrt{\frac{1+2a\rho}{4\rho}} - \int_{\sqrt{\rho}}^{\infty} \frac{1}{2} e^{-u^2} \frac{du}{u^2\sqrt{1+2au^2}}$$

and

$$\int_{\sqrt{\rho}}^{\infty} \frac{1}{2} e^{-u^2} \frac{du}{u^2\sqrt{1+2au^2}}$$

$$< \frac{1}{\rho^{3/2}\sqrt{1+2a\rho}} \int_{\sqrt{\rho}}^{\infty} \frac{1}{2} u e^{-u^2} du$$

$$= \frac{1}{2\rho(1+2a\rho)} e^{-\rho}\sqrt{\frac{1+2a\rho}{4\rho}}$$

we have for large ρ

$$N_+ \approx \frac{1}{\sqrt{\pi}} \frac{B_{IF}}{\sqrt{12}} e^{-\rho}\sqrt{\frac{1+2a\rho}{4\rho}} \tag{15}$$

which is the same as Equation (5. 14) of Reference 1. The approximate value of N_+ given by Equation (15) is higher than the exact value, but the difference is less than

$$\frac{100}{2\rho(1+2a\rho)} \quad \text{per cent.}$$

of the approximate value. By the use of Equations (12) and (14) one can write (15) as

$$N_+ \approx \frac{1}{4\pi\sqrt{\pi}} k\,\theta\, e^{-\rho} \sqrt{1+\frac{1}{3\rho}\left(m+\frac{\pi}{2\theta}\right)^2} \tag{16}$$

According to Equation (7.6) of Reference 1,

$$W_y(\omega) = 2\int_0^\infty R_{n_s}(\tau)\, e^{-[R_\varphi(0) - R_\varphi(\tau)]} \cos\omega\tau\, d\tau \tag{17}$$

where

$$R_{n_s}(\tau) = 4\,\mathcal{E}_n^2 \int_0^{\pi B_{IF}} \cos\omega\tau \,\frac{d\omega}{2\pi}$$

$$= \frac{2\,\mathcal{E}_n^2}{\pi}\,\frac{1}{\tau}\,\sin \pi B_{IF}\tau$$

$$R_{n_s}(\tau) \longrightarrow 2\,\mathcal{E}_n^2\,\delta(\tau) \quad \text{as } B_{IF} \longrightarrow \infty$$

$$R_\varphi(0) - R_\varphi(\tau) = 2\beta^2 \int_0^\infty \frac{1}{\omega^2}\, S(\omega)(1-\cos\omega\tau)\,\frac{d\omega}{2\pi}$$

$$R_\varphi(0) - R_\varphi(\tau) = 0 \quad \text{for } \tau = 0$$

$$> 0 \quad \text{for } |\tau| > 0$$

With $S(\omega) = S_a(\omega)$ as given by Equation (3),

$$R_\varphi(0) - R_\varphi(\tau) = \frac{\beta^2\mathcal{E}_a^2}{\pi}\int_0^\infty \left(\frac{2}{\omega^2}\sin^2\frac{\omega\tau}{2} - \frac{1}{\omega^2+k^2} + \frac{\cos\omega\tau}{\omega^2+k^2}\right) d\omega$$

$$= \frac{\beta^2\mathcal{E}_a^2}{\pi}\left(|\tau|\frac{\pi}{2} - \frac{1}{k}\frac{\pi}{2} + \frac{1}{k}\frac{\pi}{2}\, e^{-k|\tau|}\right)$$

$$= \frac{\beta^2\mathcal{E}_a^2}{2k}\left(k|\tau| - 1 + e^{-k|\tau|}\right)$$

and

$$W_y(\omega) = \frac{4}{\pi} \mathcal{E}_n^2 \int_0^\infty \frac{1}{\tau} e^{-\frac{\beta^2 \mathcal{E}_a^2}{2k}(k\tau - 1 + e^{-k\tau})} \sin \pi B_{IF} \tau \cos \omega\tau \, d\tau$$

$$W_y(\omega) \rightarrow 2\mathcal{E}_n^2 \quad \text{AS} \quad B_{IF} \rightarrow \infty$$

With $S(\omega) = S_b(\omega)$ as given by Equation (4),

$$R_\varphi(0) - R_\varphi(\tau) = \frac{\beta^2 \mathcal{E}_b^2}{\pi} \int_{v\frac{\pi}{2}k}^{\frac{\pi}{2}k} \frac{1}{\omega^2}(1 - \cos\omega\tau)\, d\omega$$

$$= \frac{\beta^2 \mathcal{E}_b^2}{\pi} \left\{ \frac{2}{\pi}\frac{1}{k}\frac{1-v}{v} - \frac{2}{\pi}\frac{1}{k}\left(\frac{1}{v}\cos\frac{v\pi k\tau}{2} - \cos\frac{\pi k\tau}{2}\right) + |\tau| \int_{v\frac{\pi}{2}k|\tau|}^{\frac{\pi}{2}k|\tau|} \frac{\sin x}{x}\, dx \right\}$$

$$\equiv F(|\tau|)$$

and

$$W_y(\omega) = \frac{4}{\pi} \mathcal{E}_n^2 \int_0^\infty \frac{1}{\tau} e^{-F(\tau)} \sin \pi B_{IF} \tau \cos\omega\tau \, d\tau$$

$$W_y(\omega) \longrightarrow 2\mathcal{E}_n^2 \quad \text{AS} \quad B_{IF} \longrightarrow \infty$$

Since B_{IF} is usually much wider than the bandwidth of the output filter (in our case, a Wiener filter), we will approximate

$$W_y(\omega) \approx 2\mathcal{E}_n^2, \quad -\infty < \omega < \infty \tag{18}$$

for either modulating process. This assumption simplifies the mathematics and, when used with Equations (15) or (16), yields results slightly on the conservative side. The expression (1) for the discriminator output noise power spectrum now becomes

$$N_D(\omega) \approx \frac{2\,\mathcal{E}_n^2}{\beta^2 E_0^2}\left(\psi^2 + \omega^2\right) \tag{19}$$

where

$$\psi^2 \equiv \frac{E_0^2}{\mathcal{E}_n^2}\, 4\pi^2 N_+$$

$$\approx \sqrt{\pi}\, k\, \frac{E_0^2}{\mathcal{E}_n^2}\, \theta\, e^{-\rho} \sqrt{1 + \frac{1}{3\rho}\left(m + \frac{\pi}{2\theta}\right)^2} \qquad \text{[By (16)]}$$

or, using (7) and then (12),

$$\psi^2 \approx k^2 \frac{4}{\sqrt{\pi}}\, \theta^2 \left(m + \frac{\pi}{2\theta}\right) \rho\, e^{-\rho} \sqrt{1 + \frac{1}{3\rho}\left(m + \frac{\pi}{2\theta}\right)^2} \tag{20}$$

Defining the channel quality factor λ_0 by

$$\lambda_0^4 \equiv \frac{\beta^2 E_0^2\, \mathcal{E}_a^2}{2\,\mathcal{E}_n^2\, k^2}$$

$$= \frac{E_0^2/2}{2\,\mathcal{E}_n^2\, k/2}\, \theta^2 = \frac{B_{IF}}{k/2}\, \rho\, \theta^2 = \frac{2}{\pi}\left(m + \frac{\pi}{2\theta}\right)\theta^3 \rho \tag{21}$$

finally yields

$$N_D(\omega) \approx \frac{\mathcal{E}_a^2}{k^2 \lambda_0^4}\left(\psi^2 + \omega^2\right) \tag{22}$$

We note that ψ^2, given by (20), decreases rapidly as ρ increases and its contribution to the mean square error of the output becomes negligible for ρ sufficiently large. In Chapter II, the contribution of the ψ^2 term in (22) was neglected, and the results there obtained are, consequently, applicable only for large ρ.

As ρ is decreased below a certain value, the ψ^2 term can no longer be neglected and it has the effect of depressing the output signal-to-noise ratio below the values previously computed. Further decreasing the value of ρ results in a rapid deterioration of the output signal-to-noise ratio.

2. Threshold Investigations

The threshold phenomenon will now be quantitatively investigated for the following cases:

Case I — Modulation Spectrum $S_a(\omega)$, Infinite Delay Wiener Filter

Case II — Modulation Spectrum $S_a(\omega)$, Zero Delay Wiener Filter

Case III — Modulation Spectrum $S_b(\omega)$, Infinite Delay Wiener Filter

The computational procedure used to specify the Wiener filters is the same as used in Chapter II.

Case I — Modulation Spectrum $S_a(\omega)$, Infinite Delay Wiener Filter

The frequency response of the infinite delay Wiener filter is

$$Y(\omega) = \frac{S_a(\omega)}{S_a(\omega) + N_D(\omega)} \tag{23}$$

With this filter connected to the output of the discriminator, the mean square error between the filter output and the modulating signal is

$$H_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_a(\omega)\,N_D(\omega)}{S_a(\omega) + N_D(\omega)}\, d\omega \tag{24}$$

Using Equations (3) and (22),

$$S_a(\omega) + N_D(\omega) = \frac{\mathcal{E}_a^2}{k^2\lambda_0^4}\,\frac{1}{\omega^2+k^2}\left\{b^4\lambda_0^4 + (\gamma^2+\omega^2)(\omega^2+k^2)\right\}$$

$$= \frac{\mathcal{E}_a^2}{k^2\lambda_0^4}\,\frac{1}{\omega^2+k^2}\left(\omega^4 + k_1^2\omega^2 + k_1^4\lambda_{01}^4\right) \tag{25}$$

where

$$k_1^2 \equiv k^2 \left(1 + \frac{\psi^2}{k^2}\right) \tag{26}$$

$$\lambda_{01}^4 \equiv \frac{k^4}{k_1^4}\left(\lambda_0^4 + \frac{\psi^2}{k^2}\right) \tag{27}$$

Then*

$$H_\infty = \frac{1}{2\pi} \mathcal{E}_a^2 k^2 \int_{-\infty}^{\infty} \frac{\psi^2 + \omega^2}{\omega^4 + k_1^2 \omega^2 + k_1^4 \lambda_{01}^4} \, d\omega$$

$$= \frac{1}{2\pi} \mathcal{E}_a^2 k^2 \left(\psi^2 \frac{\pi}{k_1^3} \frac{1}{\lambda_{01}^2} + \frac{\pi}{k_1}\right) \frac{1}{\sqrt{2\lambda_{01}^2 + 1}} \tag{28}$$

Using Equation (5),

$$H_\infty = \frac{k}{k_1}\left(\frac{\psi^2}{k^2} \frac{k^2/k_1^2}{\lambda_{01}^2} + 1\right) \frac{1}{\sqrt{2\lambda_{01}^2 + 1}} P$$

Using Equations (26) and (27),

$$H_\infty = \left(1 + \frac{\psi^2/k^2}{\sqrt{\lambda_0^4 + \psi^2/k^2}}\right)\left\{2\sqrt{\lambda_0^4 + \psi^2/k^2} + 1 + \psi^2/k^2\right\}^{-1/2} P$$

Hence, by defining

$$\lambda_1^4 \equiv \lambda_0^4 + \frac{\psi^2}{k^2} \tag{29}$$

$$\gamma_1 \equiv \sqrt{2\lambda_1^2 + 1 + \frac{\psi^2}{k^2}} \tag{30}$$

*See Appendix II of Chapter II for the evaluation of the integral.

the output signal power to mean square error ratio is

$$\frac{P}{H} = \frac{1}{1 + \frac{1}{\lambda_1^2}\frac{\psi^2}{k^2}} \gamma_1 \tag{31}$$

If we take $\psi = 0$, this reduces to

$$\left(\frac{P}{H_\infty}\right)_{\psi=0} = \sqrt{2\lambda_0^2 + 1} \equiv \gamma_c \tag{32}$$

which is identical to Equation (158) of Chapter II. From Equations (31) and (32),

$$R_\infty \equiv 10 \, LOG_{10} \frac{P/H_\infty}{(P/H_\infty)_{\psi=0}}$$

$$= -\left\{ 10 \, LOG_{10} \left(1 + \frac{1}{\lambda_1^2}\frac{\psi^2}{k^2}\right) - 5 \, LOG_{10} \frac{2\lambda_1^2 + 1 + \psi^2/k^2}{2\lambda_0^2 + 1} \right\} \tag{33}$$

By use of Equations (20), (21), and (29), R_∞ can be expressed as a function of ρ, m and θ.

When $\psi^2/k^2 \ll \lambda_0^2 \gg 1/2$, $\lambda_1^2 \approx \lambda_0^2$ and Equation (31) yields

$$\frac{P}{H_\infty} \approx \left(1 - \frac{1}{\lambda_0^2}\frac{\psi^2}{k^2}\right)\left(1 + \frac{1}{2}\frac{1}{2\lambda_0^2}\frac{\psi^2}{k^2}\right)\gamma_0$$

$$\approx \left(1 - \frac{3}{4}\frac{1}{\lambda_0^2}\frac{\psi^2}{k^2}\right)\gamma_0 \tag{34}$$

Threshold may be considered to occur when

$$10 \, LOG_{10} \frac{P/H_\infty}{(P/H_\infty)_{\psi=0}} = 10 \, LOG_{10} (1 - \Delta) = -\nu \; db.$$

where Δ is positive but smaller than 1, so that $\nu < 1$. When $\Delta \ll 1$, (34) applies for practical cases (where $\lambda_0^2 \gg 1/2$) so that

threshold occurs when

$$\frac{3}{4}\,\frac{1}{\lambda_o^2}\,\frac{\psi^2}{k^2} \approx \Delta \tag{35}$$

Substituting Equations (20) and (21) into (35),

$$\frac{3}{\sqrt{2}}\sqrt{\theta}\; e^{-\rho_T}\;\sqrt{\left(m+\frac{\pi}{2\theta}\right)\left\{\rho_T+\frac{1}{3}\left(m+\frac{\pi}{2\theta}\right)^2\right\}} \approx \Delta \tag{36}$$

where ρ_T indicates the value of ρ at threshold. Defining

$$b \equiv \left(1+\frac{\pi}{2m\theta}\right)\left[\frac{\rho_T}{m^2}+\frac{1}{3}\left(1+\frac{\pi}{2m\theta}\right)^2\right]$$

and taking logarithms, one obtains

$$\rho_T = \frac{1}{2\,\mathrm{LOG}\,e}\left(\mathrm{LOG}\frac{9}{2} - 2\,\mathrm{LOG}\,\Delta + 3\,\mathrm{LOG}\,m + \mathrm{LOG}\,b + \mathrm{LOG}\,\theta\right) \tag{37}$$

Thus, one can find ρ_T for given m and θ by trial. When $4 \leqslant \theta \leqslant 1024$, $m \geqslant 2$, and $\Delta \leqslant .045$ (or $\nu \leqslant 0.20$), log b varies only slightly so that ρ_T varies approximately as log θ for fixed m.

Case II <u>Modulation Spectrum $S_a(\omega)$, Zero Delay Wiener Filter</u>

In order to define the transfer function of the required Wiener filter, we first find a realizable frequency response $Y_1(\omega)$ which satisfies

$$Y_1(\omega)\, Y_1^*(\omega) = \frac{1}{S_a(\omega) + N_D(\omega)}$$

Substituting (25) in this,

$$Y_1(\omega)\, Y_1^*(\omega) = \frac{k^2\lambda_0^4}{\mathcal{E}_a^2}\;\frac{\omega^2+k^2}{\omega^4+k_1^2\,\omega^2+k_1^4\,\lambda_{01}^4}$$

Therefore,

$$Y_1(\omega) = \frac{k\lambda_0^2}{\mathcal{E}_a}\;\frac{-i\,(\omega - ik)}{(\omega - B_2 - iB_1)(\omega + B_2 - iB_1)}$$

where

$$B_1 = \frac{k_1}{2}\sqrt{2\lambda_{01}^2+1}\ , \qquad B_2 = \frac{k_1}{2}\sqrt{2\lambda_{01}^2-1} \tag{38}$$

We next write

$$Y_2(\omega) = \frac{S_a(\omega)}{S_a(\omega)+N_D(\omega)}\ \frac{1}{Y_1(\omega)}$$

$$= i\,\mathcal{E}_a k^3 \lambda_0^2\ \frac{1}{(\omega-ik)(\omega-B^2+iB_1)(\omega+B_2+iB_1)}$$

$$= i\,\mathcal{E}_a k^3 \lambda_0^2 \left(\frac{a_1}{\omega-ik} + \frac{a_2}{\omega-B_2+iB_1} + \frac{a_3}{\omega+B_2+iB_1}\right)$$

where

$$a_1 = \frac{1}{(B_2-iB_1-ik)(-B_2-iB_1-ik)} = -\frac{1}{k^2(1+\lambda_1^2+\gamma_1)}$$

$$a_2 = \frac{1}{2B_2(B_2-iB_1-ik)} = -a_1\ \frac{B_2+iB_1+ik}{2B_2}$$

$$a_3 = \frac{1}{2B_2(B_2+iB_1+ik)} = -a_1\ \frac{B_2-iB_1-ik}{2B_2}$$

Since

$$\int_0^\infty (i\,e^{-kt})\,e^{-i\omega t}\,dt = \frac{1}{\omega-ik}$$

and

$$\int_{-\infty}^0 (-i\,e^{iAt})\,e^{-i\omega t}\,dt = \frac{1}{\omega-A} \qquad (A = \pm B_2 - iB_1)$$

the impulse response corresponding to the frequency response $Y_2(\omega)$ for $t \geqslant 0$ is

$$K_2(t) = i\,\mathcal{E}_a k^3 \lambda_0^2\, a_1\,(i\,e^{-kt}) = \frac{\mathcal{E}_a k \lambda_0^2}{1+\lambda_1^2+\gamma_1}\ e^{-kt}$$

Our next step is to find the frequency response $Y_3(\omega)$ corresponding to the impulse response

$$K_3(t) = \begin{cases} K_2(t) \text{ FOR } t \geq 0 \\ 0 \quad \text{FOR } t < 0 \end{cases}$$

Thus,

$$Y_3(\omega) = \frac{\mathcal{E}_a k \lambda_0^2}{1+\lambda_1^2+\gamma_1} \int_0^{\infty} e^{-kt} e^{-i\omega t} dt = \frac{\mathcal{E}_a k \lambda_0^2}{1+\lambda_1^2+\gamma_1} \; \frac{1}{i(\omega - ik)}$$

Finally, the frequency response of the required zero delay Wiener filter is

$$Y_4(\omega) = Y_3(\omega)\, Y_1(\omega)$$

$$= -\frac{k^2 \lambda_0^4}{1+\lambda_1^2+\gamma_1} \; \frac{1}{(\omega - B^2 - i B_1)(\omega + B_2 - i B_1)}$$

$$= \frac{k^2 \lambda_0^4}{1+\lambda_1^2+\gamma_1} \; \frac{1}{-\omega^2 + 2 B_1 i \omega + k_1^2 \lambda_{01}^2} \tag{39}$$

With this filter connected to the output of the discriminator, the output N_0 due to noise is

$$N_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} N_D(\omega) \left| Y_4(\omega) \right|^2 d\omega$$

$$= \frac{1}{2\pi} \; \frac{\mathcal{E}_a^2 k^2 \lambda_0^4}{(1+\lambda_1^2+\gamma_1)^2} \int_{-\infty}^{\infty} \frac{\psi^2 + \omega^2}{\omega_0^4 + k_1^2 \omega^2 + k_1^4 \lambda_{01}^4} \, d\omega \tag{40}$$

By comparing this with (28) and then using (31),

$$N_0 = \frac{\lambda_0^4}{(1+\lambda_1^2+\gamma_1)^2} H_\infty$$

$$= \frac{\lambda_0^4}{(1+\lambda_1^2+\gamma_1)^2} \left(1 + \frac{1}{\lambda_1^2} \frac{\psi^2}{k^2}\right) \frac{1}{\gamma_1} P \tag{41}$$

The contribution to the mean square error of the output due to distortion is

$$D_0 = \frac{1}{2\pi} \mathcal{E}_a^2 k^2 \int_{-\infty}^{\infty} \frac{1}{\omega^2+k^2} \left| 1 - Y_4(\omega) \right|^2 d\omega \tag{42}$$

Since

$$\left| 1 - Y_4(\omega) \right|^2 = 1 + \frac{k^4 \lambda_0^8}{(1+\lambda_1^2+\gamma_1)^2} \frac{1}{\omega^4 + k_1^2 \omega^2 + k_1^4 \lambda_{01}^4} - \frac{k^2 \lambda_0^4}{1+\lambda_1^2+\gamma_1} \frac{2(-\omega^2 + k_1^2 \lambda_{01}^2)}{\omega^4 + k_1^2 \omega^2 + k_1^4 \lambda_{01}^4}$$

$$= 1 + k^2 \frac{2\lambda_0^4}{1+\lambda_1^2+\gamma_1} \frac{\omega^2}{\omega^4 + k_1 \omega^2 + k^4 \lambda_{01}^4} - k^4 \frac{\lambda_0^4}{1+\lambda_1^2+\gamma_1} \left(2 \frac{k_1^2 \lambda_{01}^2}{k^2} - \frac{\lambda_0^4}{1+\lambda_1^2+\gamma_1} \right) \frac{1}{\omega^4 + k_1^2 \omega^2 + k_1^4 \lambda_{01}^4}$$

$$D_0 = \frac{1}{2\pi} \mathcal{E}_a^2 k^2 \left\{ \frac{\pi}{k} + k^2 \frac{2\lambda_0^4}{1+\lambda_1^2+\gamma_1} C_2 - k^4 \frac{\lambda_0^4}{1+\lambda_1^2+\gamma_1} \left(2 \frac{k_1^2 \lambda_{01}^2}{k^2} - \frac{\lambda_0^4}{1+\lambda_1^2+\gamma_1} \right) C_0 \right\} \tag{43}$$

where

$$C_2 = \int_{-\infty}^{\infty} \frac{\omega^2}{(\omega^2+k^2)(\omega^4 + k_1^2 \omega^2 + k_1^4 \lambda_{01}^4)} d\omega$$

$$= 2\pi i \left[\frac{-k^2}{i\, 2k(k_1^4 \cdot \lambda_{01}^4)} + \frac{1}{i\, 8 B_1 B_2} \left\{ \frac{B_2 + iB_1}{(B_2 + iB_1)^2 + k^2} - \frac{-B_2 + iB_1}{(-B_2 + iB_1)^2 + k^2} \right\} \right]$$

$$= \frac{\pi}{k^3}\left\{ \frac{k/k_1}{2\,\lambda_{01}^2 + 1} \; \frac{\lambda_{01}^2\, k_1^2/k^2 + 1}{\lambda_{01}^4\, k_1^4/k^4 - (k_1^2/k^2 - 1)} - \frac{1}{\lambda_{01}^4\, k_1^4/k^4} \right\} \tag{44}$$

and

$$C_0 = \int_{-\infty}^{\infty} \frac{1}{(\omega^2 + k^2)(\omega^4 + k_1^2\,\omega^2 + k_1^4\,\lambda_{01}^4)}\, d\omega$$

$$= \pi\left[\frac{1}{k\,k_1^4\,\lambda_{01}^4} + \frac{1}{4B_1B_2}\left\{ \frac{(B_2 + iB_1)^{-1}}{k^2 - k_1^2/2 + i\,2B_1B_2} + \frac{(B_2 - iB_1)^{-1}}{k^2 - k_1/2 - i\,2B_1B_2} \right\} \right]$$

$$= \frac{\pi}{k^5}\;\frac{1}{\lambda_{01}^2\, k_1^2/k^2}\left\{ \frac{1}{\lambda_{01}^2\, k_1^2/k^2} \right.$$

$$\left. - \frac{k/k_1}{\sqrt{2\,\lambda_{01}^2 + 1}}\;\frac{\lambda_{01}^2\, k_1^2/k^2 + (k_1^2/k - 1)}{\lambda_{01}^4\, k_1^4/k^4 - (k_1^2/k^2 - 1)} \right\} \tag{45}$$

Using the definitions of k_1, λ_{01}, λ_1, and γ_1, one finds

$$\left.\begin{aligned} \frac{k_1}{k}\sqrt{2\,\lambda_{01}^2 + 1} &= \gamma_1 \\ \lambda_{01}^2\,\frac{k_1^2}{k^2} &= \lambda_1^2 \\ \frac{k_1^2}{k^2} - 1 &= \frac{\psi^2}{k^2} \\ \frac{\lambda_{01}^4\,k_1^4}{k^4} - \left(\frac{k_1^2}{k^2} - 1\right) &= \lambda_0^4 \end{aligned}\right\} \tag{46}$$

so that

$$C_2 = \frac{\pi}{k^3}\left(\frac{1}{\gamma_1}\;\frac{\lambda_1^2 + 1}{\lambda_0^4} - \frac{1}{\lambda_1^4} \right) \tag{47}$$

$$C_0 = \frac{\pi}{k^5} \frac{1}{\lambda_1^2} \left(\frac{1}{\lambda_1^2} - \frac{1}{\gamma_1} \frac{\lambda_1^2 + \psi^2/k^2}{\lambda_0^4} \right) \tag{48}$$

Using these values of C_2 and C_0 as well as $\mathcal{E}_a^2\, k/2 = P$, the expression of D_0, Equation (43) becomes

$$D_0 = P\Bigg[1 + \frac{2}{1+\lambda_1^2+\gamma_1} \left(\frac{\lambda_1^2+1}{\gamma_1} - \frac{\lambda_0^4}{\lambda_1^4} \right)$$
$$- \frac{1}{1+\lambda_1^2+\gamma_1} \left(2\lambda_1^2 - \frac{\lambda_0^4}{1+\lambda_1^2+\gamma_1} \right) \left(\frac{\lambda_0^4}{\lambda_1^4} - \frac{\lambda_1^2+\psi^2/k^2}{\lambda_1^2} \frac{1}{\gamma_1} \right) \Bigg\}$$

or

$$\frac{D_0}{P} = 1 - \frac{\lambda_0^4}{(1+\lambda_1^2+\gamma_1)^2} \frac{\lambda_1^2+\psi^2/k^2}{\lambda_1^2} \frac{1}{\gamma_1}$$
$$- \frac{1}{1+\lambda_1^2+\gamma_1} \left\{ \frac{\lambda_0^4}{\lambda_1^4} \left(2\lambda_1^2 - \frac{\lambda_0^4}{1+\lambda_1^2+\gamma_1} + 2 \right) - 2\gamma_1 \right\}$$
$$= 1 - \frac{N_0}{P} - \frac{1}{1+\lambda_1^2+\gamma_1} \left\{ \frac{\lambda_0^4}{\lambda_1^4} \left(2\lambda_1^2 - \frac{\lambda_0^4}{1+\lambda_1^2+\gamma_1} + 2 \right) - 2\gamma_1 \right\} \tag{49}$$

Hence, the total mean square error of the output, H_0, obeys

$$\frac{H_0}{P} = \frac{D_0}{P} = \frac{N_0}{P}$$
$$= \frac{1}{1+\lambda_1^2+\gamma_1} \left\{ 1 + \lambda_1^2 + 3\gamma_1 - \frac{\lambda_0^4}{\lambda_1^4} \left(2\lambda_1^2 - \frac{\lambda_0^4}{1+\lambda_1^2+\gamma_1} + 2 \right) \right\}$$
$$= \frac{1}{1+\lambda_1^2+\gamma_1} \left[\left(3 - \frac{\lambda_0^4}{\lambda_1^4} \right) \gamma_1 + \frac{1+\lambda_1^2}{\lambda_1^4} \frac{\psi^2}{k^2} \right\}$$

so that the ratio of signal power to mean square error

$$\frac{P}{H_0} = \frac{\lambda_1^2 + \gamma_1 + 1}{\left(3 - \frac{\lambda_0^4}{\lambda_1^4}\right)\gamma_1 + \frac{\lambda_1^2 + 1}{\lambda_1^4}\,\frac{\psi^2}{k^2}} \tag{50}$$

If we take $\psi = 0$, then $\lambda_1 = \lambda_0$, $\gamma = \sqrt{2\lambda_0^2 + 1} \equiv \gamma_0$, and the above formula reduces to

$$\left(\frac{P}{H_0}\right)_{\psi=0} = \frac{\lambda_0^2 + \gamma_0 + 1}{2\gamma_0} = \frac{(\gamma_0 + 1)^2}{4\gamma_0} \tag{51}$$

which is identical to Equation (180) of Chapter II. Thus, we have

$$\begin{aligned} R_0 &\equiv 10 \log_{10} \frac{P/H_0}{(P/H_0)_{\psi=0}} \\ &= 10 \log_{10} \frac{\lambda_1^2 + \gamma_1 + 1}{\lambda_0^2 + \gamma_0 + 1} - 10 \log_{10} \frac{\left(3 - \frac{\lambda_0^4}{\lambda_1^4}\right)\gamma_1 + \frac{\lambda_1^2 + 1}{\lambda_1^4}\,\frac{\psi^2}{k^2}}{2\gamma_0} \end{aligned} \tag{52}$$

so that, by using (20), (21), (29) and $\gamma_0 \equiv \sqrt{2\lambda_0^2 + 1}$, one can calculate R_0 for various ρ and given m and θ.

Case III <u>Modulation Spectrum $S_b(\omega)$, Infinite Delay Wiener Filter</u>

The power spectrum $S_b(\omega)$ is given by (4), and the required frequency response of the infinite delay Wiener filter is for this case

$$Y(\omega) = \frac{S_b(\omega)}{S_b(\omega) + N_D(\omega)}$$

or, by (4) and (22),

$$Y(\omega) = \begin{cases} \dfrac{k^2 \lambda_0^4}{k^2 \lambda_0^4 + (1 - v)(\psi^2 + \omega^2)} & \text{FOR } v\frac{\pi}{2}k \le |\omega| \le \frac{\pi}{2}k \\ 0 \text{ ELSEWHERE} & \end{cases} \tag{53}$$

The mean square error between the filter output and the modulating function is then

$$H_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} N_D(\omega)\, Y(\omega)\, d\omega$$

$$= \frac{\mathcal{E}_a^2}{2\pi} 2 \int_{v\frac{\pi}{2}k}^{\frac{\pi}{2}k} \frac{\psi^2+\omega^2}{k^2\lambda_o^4+(1-v)(\psi^2+\omega^2)} d\omega$$

$$= \frac{\mathcal{E}_a^2}{\pi} \int_{v\frac{\pi}{2}k}^{\frac{\pi}{2}k} \left\{ \frac{1}{1-v} - \frac{k^2\lambda_o^4}{1-v} \frac{1}{k^2\lambda_o^4+(1-v)(\psi^2+\omega^2)} \right\} d\omega$$

$$= \frac{\mathcal{E}_a^2}{\pi} \left\{ \frac{\pi}{2}k - \frac{k^2\lambda_o^4/(1-v)^2}{\sqrt{\frac{1}{1-v}k^2\lambda_o^4+\psi^2}} \left(TAN^{-1} \frac{\frac{\pi}{2}k}{\sqrt{\frac{1}{1-v}k^2\lambda_o^4+\psi^2}} \right.\right.$$

$$\left.\left. - TAN^{-1} \frac{v\frac{\pi}{2}k}{\sqrt{\frac{1}{1-v}k^2\lambda_o^4+\psi^2}} \right) \right\}$$

Let $(A-B) = TAN^{-1}\zeta$ and using $\zeta = TAN(A-B) = \dfrac{TAN\ A - TAN\ B}{TAN\ A \cdot TAN\ B + 1}$,

$$H_\infty = P\left\{ 1 - \frac{2/\pi}{(1-v)^{3/2}} \frac{\lambda_o^4}{\sqrt{\lambda_o^4+(1-v)\psi^2/k^2}} TAN^{-1}\zeta \right\} \tag{54}$$

where

$$\zeta = \frac{\frac{\pi}{2}(1-v)^{3/2}\sqrt{\lambda_o^4+(1-v)\psi^2/k^2}}{\lambda_o^4+(1-v)\psi^2/k^2+(1-v)v\pi^2/4} \tag{55}$$

Define

$$\lambda_2^4 \equiv \lambda_o^4 + (1-v)\frac{\psi^2}{k^2} \tag{56}$$

Then, (55) and (54) can be written as

$$\zeta = \frac{\pi}{2}(1-v)^{3/2} \frac{1}{\lambda_2^2} \frac{1}{1+\frac{1}{4}\pi^2(1-v)v/\lambda_2^4} \tag{57}$$

and

$$\frac{P}{H_\infty} = \frac{\lambda_2^4 + \frac{\pi^2}{4}(1-v)v}{(1-v)\frac{\psi^2}{k^2} + \frac{\pi^2}{4}(1-v)v + \lambda_o^4\left(1 - \frac{TAN^{-1}\zeta}{\zeta}\right)} \tag{58}$$

If $\psi = 0$, then $\lambda_2 = \lambda_0$ and (58) reduces to

$$\left(\frac{P}{H_\infty}\right)_{\psi=0} = \frac{\lambda_0^4 + \frac{\pi^2}{4}(1-\nu)\nu}{\frac{\pi^2}{4}(1-\nu)\nu + \lambda_0^4\left(1 - \frac{\tan^{-1}\zeta_0}{\zeta_0}\right)} \tag{59}$$

where

$$\zeta_0 = \frac{\pi}{2}(1-\nu)^{3/2}\ \frac{1}{\lambda_0^2}\ \frac{1}{1+\frac{1}{4}\pi^2(1-\nu)\nu/\lambda_0^4} \tag{60}$$

When $\lambda_0^2 \geq 30$ (which holds for most cases of practical interest), $\zeta < \pi/60$, $\zeta_0 \leq \pi/60$, and

$$\frac{\pi^2}{4}(1-\nu)\nu/\lambda_2^4 < \frac{\pi^2}{4}(1-\nu)\nu/\lambda_0^4 \leq \frac{\pi}{16\lambda_0^4} \leq \frac{\pi^2}{14{,}400} \approx \frac{1}{1{,}460}$$

so that

$$\frac{\tan^{-1}\zeta}{\zeta} \approx 1 - \frac{\pi^2}{12}\ \frac{(1-\nu)^3}{\lambda_2^4}, \quad \frac{\tan^{-1}\zeta_0}{\zeta_0} \approx 1 - \frac{\pi^2}{12}\ \frac{(1-\nu)^3}{\lambda_0^4}$$

and (58) and (59) reduce to

$$\frac{P}{H_\infty} \approx \frac{\lambda_2^4/(1-\nu)}{\frac{\psi^2}{k^2} + \frac{\lambda_0^4}{\lambda_2^4}\ \frac{\pi^2}{12}(1-\nu)^2 + \frac{\pi^2}{4}\nu} \tag{61}$$

$$\left(\frac{P}{H_\infty}\right)_{\psi=0} \approx \frac{12}{\pi^2}\ \frac{1}{1-\nu^3}\ \lambda_0^4 \tag{62}$$

Then

$$\frac{P/H_\infty}{(P/H_\infty)_{\psi=0}} \approx \frac{\lambda_2^4}{\lambda_0^4}\ \frac{1+\nu+\nu^2}{\frac{12}{\pi^2}\ \frac{\psi^2}{k^2} + \frac{\lambda_0^4}{\lambda_2^4}(1-\nu)^2 + 3\nu} \tag{63}$$

Threshold occurs when this ratio decreases to $(1-\Delta)$, Δ being a positive number much smaller than 1. Thus, at threshold $\psi^2/k^2 \ll \lambda_0^4$ so that $\lambda_2^4/\lambda_0^4 \approx 1$ and threshold occurs when

$$\frac{1}{1 + \frac{12}{\pi^2}\ \frac{1}{1+\nu+\nu^2}\ \frac{\psi^2}{k^2}} \approx 1 - \Delta$$

or

$$\frac{\psi^2}{k^2} \approx \frac{\pi^2}{12}(1+\nu+\nu^2)\frac{\Delta}{1-\Delta} \tag{64}$$

Using this relation together with (20), which expresses ψ^2/k^2 in terms of ρ, m and θ, one can determine ρ_T, the value of ρ at threshold, as a function of m and θ by trial or graphical methods. We note that (64) would be exact, if

$$Y(\omega) = \begin{cases} 1 & \text{FOR } \nu \frac{\pi}{2} k \leq |\omega| \leq \frac{\pi}{2} k \\ 0 & \text{ELSEWHERE} \end{cases}$$

instead of (53) were to be used as the frequency response of the output filter.

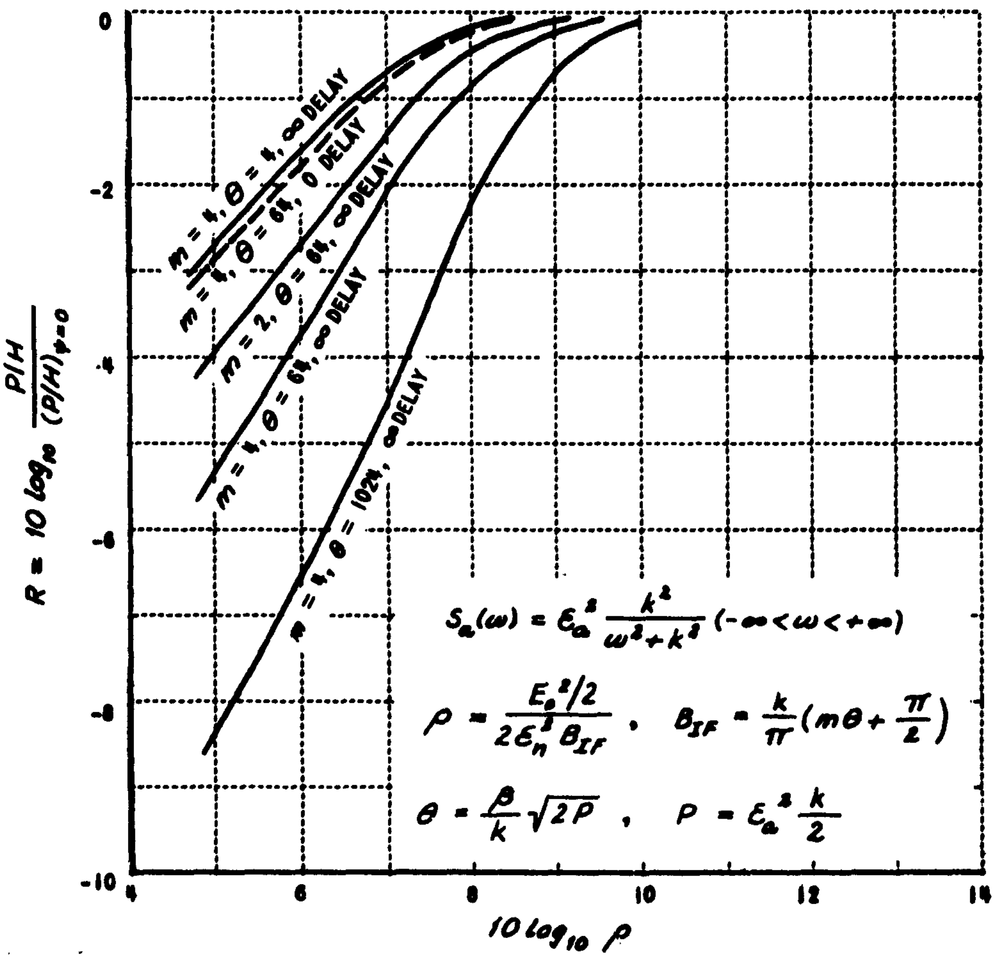
3. Discussion of Results

The main purpose of this investigation is to investigate the behavior of the output of an FM receiver in response to a stochastically modulated signal in the threshold region. In Figure 1, the decrease in the ratio of the power of the modulating signal to the mean square difference between the output and modulating signal below that predicted by the large carrier-to-noise power ratio theory has been plotted. For the range of parameters illustrated, it is evident that the deviation is negligible for values of ρ, the ratio of carrier power to the noise power in the I. F. bandwidth, greater than about 11 db, and that the threshold occurs in the region of ρ between 7 and 10 db. Recalling that the I. F. bandwidth,

$$B_{IF} = \frac{k}{\pi}\left(m\theta + \frac{\pi}{2}\right)$$

which for large $m\theta$ is proportional to $m\theta$, one notes from the figure that the deviation from the large carrier-to-noise power ratio theory is more rapid the greater the I. F. bandwidth, as one would expect. The apparently gentler behavior of the $m = 4$, $\theta = 64$, 0 delay curve compared to the $m = 4$, $\theta = 64$, ∞ delay curve is explainable as follows. Since these cases both have the same I. F. bandwidth, the discriminator outputs are identical. However, the zero delay Wiener filter results in approximately 6 db more noise output power than the infinite delay Wiener filter above threshold. For any rate of occurrence of impulses ($N_+ + N_-$ times per second) or for any value of ψ, the relative (db) increase in noise output power is less when the zero delay filter is used than when the infinite delay filter is used*. On the absolute basis, the infinite delay filter will always give superior performance. This phenomenon is more clearly demonstrated by Figure 2 where P/H and $(P/H)_{\psi=0}$ are plotted against the channel quality factor λ_0^4. The channel quality factor λ_0^4 may be

*Because the ratio of the noise equivalent bandwidth of the zero delay filter to that of the infinite delay filter is less than 6 db.



0
-2
-4
-6
-8
-10
4
6
8
10
12
14
$R = 10 \log_{10} \frac{P/H}{(P/H)_{\phi=0}}$
$10 \log_{10} \rho$
m = 4, θ = 4, ∞ DELAY
m = 4, θ = 64, 0 DELAY
m = 2, θ = 64, ∞ DELAY
m = 4, θ = 64, ∞ DELAY
m = 4, θ = 1024, ∞ DELAY
$S_a(\omega) = \mathcal{E}_a^2 \frac{k^2}{\omega^2 + k^2} \; (-\infty < \omega < +\infty)$
$\rho = \frac{E_0^2/2}{2\mathcal{E}_n^2 B_{IF}}, \quad B_{IF} = \frac{k}{\pi}\left(m\theta + \frac{\pi}{2}\right)$
$\theta = \frac{\beta}{k}\sqrt{2P}, \quad P = \mathcal{E}_a^2 \frac{k}{2}$

Fig. 1

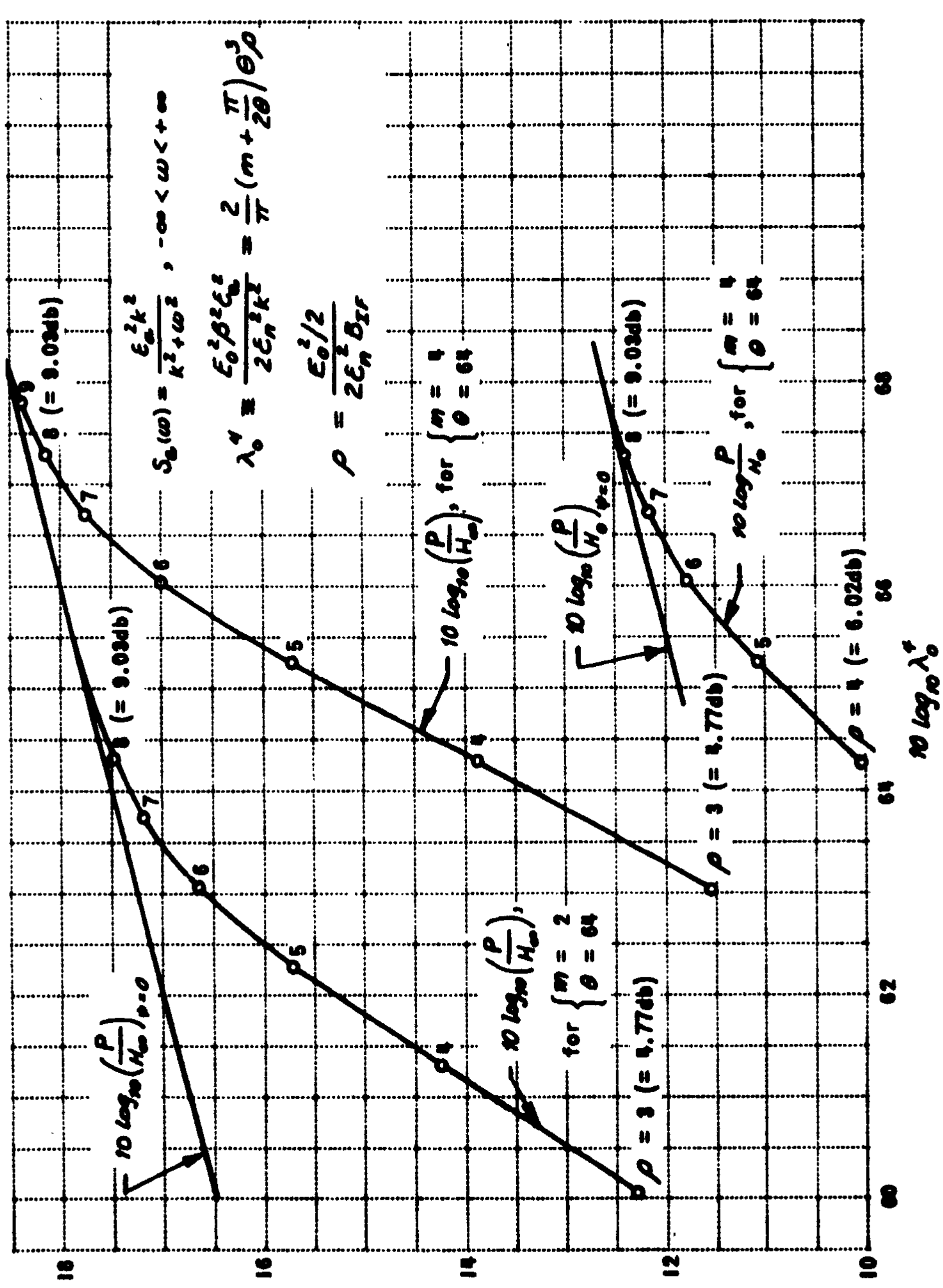

$S_e(\omega) = \frac{E_e^2 k^2}{k^2+\omega^2}$, $-\infty < \omega < +\infty$
$\lambda_0^4 \equiv \frac{E_0^2 \beta^2 E_e^2}{2 E_n^2 k^2} = \frac{2}{\pi}\left(m + \frac{\pi}{2\theta}\right)\theta^3 \rho$
$\rho = \frac{E_0^2/2}{2 E_n^2 B_{IF}}$
$10 \log_{10}\left(\frac{P}{H_\omega}\right)_{\theta=0}$
$10 \log_{10}\left(\frac{P}{H_\omega}\right)$, for $m = 2$, $\theta = 64$
$\rho = 3$ (= 4.77db)
8 (= 9.03db)
$10 \log_{10}\left(\frac{P}{H_\omega}\right)$, for $m = 4$, $\theta = 64$
$\rho = 3$ (= 4.77db)
8 (= 9.03db)
$10 \log_{10}\left(\frac{P}{H_o}\right)_{\theta=0}$
$10 \log \frac{P}{H_o}$, for $m = 4$, $\theta = 64$
$\rho = 4$ (= 6.02db)
8 (= 9.03db)
10
12
14
16
18
60
62
64
66
68
$10 \log_{10} \lambda_0^4$

Fig. 2

expressed in several ways in terms of the other parameters of the system. Thus,

$$\lambda_o^4 = \frac{E_o^2 \beta^2 \mathcal{E}_a^2}{2 \mathcal{E}_n^2 k^2}$$

$$= \frac{E_o^2/2}{2\mathcal{E}_n^2 k/2} \frac{\beta^2}{k^2} P$$

$$= \frac{E_o^2/2}{2\mathcal{E}_n^2 k/2} \frac{\theta^2}{2}$$

$$= \frac{B_{IF}}{k/2} \rho \theta^2$$

$$= \frac{2}{\pi}\left(m + \frac{\pi}{2\theta}\right)\theta^3 \rho$$

where

$$\theta = \frac{\beta}{k}\sqrt{2P} = \frac{\beta}{k}\sqrt{\mathcal{E}_a^2 k} \quad \text{.... modulation parameter}$$

$$\rho = \frac{E_o^2/2}{2\mathcal{E}_n^2 B_{IF}} \quad \text{.... carrier-to-noise (in I. F. bandwidth } B_{IF}\text{) power ratio}$$

$$B_{IF} = \frac{k}{\pi} m\theta + \frac{k}{2} \quad \text{.... I. F. bandwidth}$$

These formulas permit ready conversion of plots of P/H versus λ_o^4 against other parameters, e. g., to convert from λ_o^4 db to I. F. carrier-to-noise ratio, ρ db subtract 10 log$\left[\frac{2}{\pi}(m + \pi/2\theta)\,\theta^3\right]$. From the first expression for λ_o^4, it will be noted that λ_o^4 is independent of m, the parameter which established the I. F. bandwidth, and hence for fixed modulation (i. e., for fixed β, $\mathcal{E}_a^2$ and k), the channel quality factor λ_o^4 is proportional to the ratio of carrier power to noise power density.

Figure 3 illustrates performance near threshold for band-limited modulating signals with two different power spectra, corresponding to $\nu = 0$ and $\nu = 1/2$. The two power spectra are shown in the insert of the figure. It might be mentioned that the curves shown in Figure 3 are practically unchanged if a sharp cut-off filter, which has frequency response

$$Y_c(\omega) = \begin{cases} 1 & \text{FOR } \nu\frac{\pi}{2}k \leq |\omega| \leq \frac{\pi}{2}k \\ 0 & \text{ELSE WHERE} \end{cases}$$

is used instead of the Wiener filter.

REFERENCES

1. Rice, S. O., "Noise in FM Receivers", Bell Telephone Laboratory Memo, MM-62-3241-5, April 25, 1962.

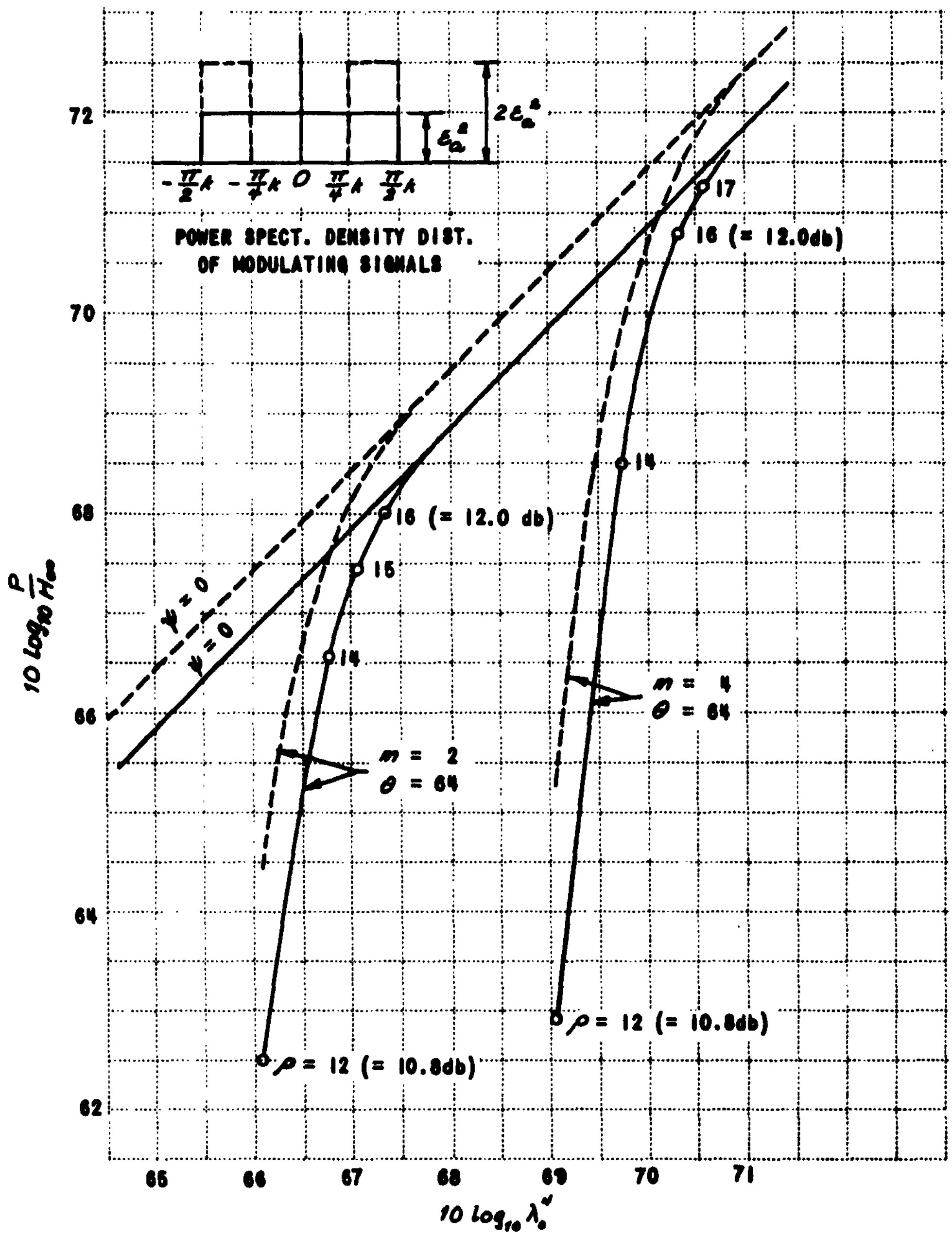

POWER SPECT. DENSITY DIST. OF MODULATING SIGNALS
72
70
68
66
64
62
65
66
67
68
69
70
71
ψ = 0
ψ = 0
16 (= 12.0 db)
15
14
m = 2
θ = 64
ρ = 12 (= 10.8db)
17
16 (= 12.0db)
14
m = 4
θ = 64
ρ = 12 (= 10.8db)

Fig. 3

IV

APPLICATION OF INFORMATION THEORY TO BOUND THE PERFORMANCE OF COMMUNICATIONS SYSTEMS

A question of continuing interest is to what extent the theorems of information theory can be applied in order to establish bounds on the attainable performance of communications systems. Although a simple or complete answer to this question cannot be given, some preliminary results which we have obtained are presented. In the process of doing this, we also hope to be able to dispel some widely held misconceptions.

As background material, we will very briefly review some of the terminology, theorems and results of information theory. The proofs of the theorems and derivations of the results can be found in the references which are given.

We shall assume that all processes with which we are concerned are ergodic.* The most important properties of an ergodic process for our purposes are: that any sample function of the process observed over a sufficiently long time exhibits a behavior typical of the process, and that time and ensemble statistics are identical. In the case of a discrete random process, such as a sequence of digital data, only a finite number $N(T)$ of sequences having duration T seconds have a nonvanishing probability. The rate of generation of information of such a process is defined as

$$R = \lim_{T \to \infty} \frac{\ln N(T)}{T} \qquad (1)$$

The capacity C of a channel is defined as the maximum rate of transmission of information of which the channel is capable. The fundamental theorem of information theory states** that it is possible to transmit information at a rate $R \leq C$ with arbitrarily small probability of error, but that this is impossible if $R > C$. In order to achieve rates very close to channel capacity, very lengthy codes (the explicit construction of which is not known in general) may have to be used. The fundamental theorem applies to continuous as well as to discrete channels.

---

* The properties of an ergodic process are discussed in Reference 1 (pp. 15, 57) and Reference 2 (pp. 67-68).

** Reference 1 (pp. 39, 67)

Consider a channel in which the transmitted signal x is perturbed by an additive noise n so that the channel output y is given by $y = x + n$. Further, let the bandwidth of this channel be restricted to W cps and the mean square value of the input $\langle x^2 \rangle = P$. Then the capacity of the channel is bounded by

$$W \ln \frac{P+N_1}{N_1} \leq C \leq W \ln \frac{P+N}{N_1} \tag{2*}$$

where N = average power of the noise
N_1 = entropy power of the noise

The entropy power N_1 of a random process is a measure of the randomness of the process. White gaussian noise has the greatest entropy (randomness) for a given power and bandwidth of all random processes. Entropy power of any process is defined as the power of a white gaussian noise having the same bandwidth and entropy as the process under consideration. Therefore, we find that for white gaussian noise, the entropy power N_1 is equal to the actual power N and, for any other process, the entropy power is less than the actual power. If the additive noise is white and gaussian, the upper and lower bounds in Equation (2) are identical so that

$$C = W \ln \left(1 + \frac{P}{N}\right) \tag{3}$$

which is without a doubt the best known equation of information theory.

In analog communications systems, one is interested in reproducing a continuous waveform presented to the input of the system at the output. Since a continuously variable waveform can take on an infinite number of values, its exact transmission would require a channel of infinite capacity. In practice, one is not interested in reconstructing a continuously varying waveform exactly, but may instead decide that the communication system is satisfactory, provided that the mean square error between output and input does not exceed some specified value, say N_c. N_c may therefore be called a mean square error fidelity criterion. Satisfactory communication can then be obtained by transmitting, instead of the actual waveform produced by the source, one of a number $N(T)$ of preselected sample functions of duration T, the sample functions being selected so that the mean square difference between the sample function and the actual waveform is less than N_c. The rate of generation of information is then given by

$$R = \lim_{T \to \infty} \frac{\ln N(T)}{T} \tag{4}$$

* Reference 1 (p. 68)

where $N(T)$ is the minimum number of sample functions required to satisfy the fidelity criterion $N_{\mathcal{L}}$. This rate is bound by*

$$W_S \ln \frac{P_1}{N_{\mathcal{L}}} \leq R \leq W_S \ln \frac{P}{N_{\mathcal{L}}} \tag{5}$$

where P = power of the source

P_1 = entropy power of the source

$N_{\mathcal{L}}$ = permissible mean square error

W_S = bandwidth of source

By the fundamental theorem of information theory, it is then possible to transmit continuous information over a channel of capacity C with a mean square error not exceeding $N_{\mathcal{L}}$ provided that $C \geq R$ where R is given by Equation (5). If the source has the statistics of a white gaussian noise process then the upper and lower bounds of Equation (5) are identical and

$$R = W_S \ln \frac{P}{N_{\mathcal{L}}} \tag{6}$$

Suppose now that the communications channel has a bandwidth W_{CH} , signal power P_{CH} and is perturbed by additive white gaussian noise of intensity N_0 watts/cps. The capacity of this channel is then

$$C_{CH} = W_{CH} \ln \left(1 + \frac{P_{CH}}{N_0 W_{CH}}\right) \tag{7}$$

The information rate R which can be transmitted over this channel is then

$$R \leq C_{CH} \tag{8}$$

Substituting Equations (6), (7), we find

$$W_S \ln \frac{P}{N_{\mathcal{L}}} \leq W_{CH} \ln \left(1 + \frac{P_{CH}}{N_0 W_{CH}}\right) \tag{9}$$

which may be solved for $\frac{P}{N_{\mathcal{L}}}$ to yield

---

* Reference 1 (p. 80)

$$
\begin{aligned}
\frac{P}{N_c} &\leq \left(1 + \frac{P_{CH}}{N_0 W_S} \frac{W_S}{W_{CH}}\right)^{\frac{W_{CH}}{W_S}} \\
&= \left(1 + \frac{P_{CH}}{N_0 W_S} \cdot \frac{1}{n}\right)^{n} \\
\lim_{n \to \infty} \frac{P}{N_c} &\leq e^{\frac{P_{CH}}{N_0 W_S}}
\end{aligned}
\tag{10}
$$

with $n = \frac{W_{CH}}{W_S}$ = bandwidth expansion factor.

It should be carefully born in mind that Equation (10) applies only to the case where the source has the statistics of a white gaussian noise (since we assumed $P_1 = P$) and that N_c is defined as the mean square error between the output and input waveforms.

If the input source does not have the statistics of white gaussian noise, then

$$
R < W_S \ln \frac{P}{N_c} \tag{11}
$$

Satisfactory transmission requires $C \geq R$ so that

$$
\begin{aligned}
W_S \ln \frac{P}{N_c} > R \leq C = W_{CH} \ln \left(1 + \frac{P_{CH}}{N_0 W_{CH}}\right) \\
\frac{P}{N_c} \not\geq \left(1 + \frac{P_{CH}}{N_0 W_S} \frac{1}{n}\right)^{n}
\end{aligned}
\tag{12}
$$

From (12) it is clear that this approach will not yield a generally valid bound on the maximum value of P/N_c which can be obtained by use of a given channel.* On the other hand, using the lower bound of Equations (5) and (8), we find

$$
\begin{aligned}
W_S \ln \frac{P_1}{N_c} \leq R \leq C = W_{CH} \ln \left(1 + \frac{P_{CH}}{N_0 W_{CH}}\right) \\
\frac{P_1}{N_c} \leq \left(1 + \frac{P_{CH}}{N_0 W_S} \frac{1}{n}\right)^{n} \\
\lim_{n \to \infty} \frac{P_1}{N_c} \leq e^{\frac{P_{CH}}{N_0 W_S}}
\end{aligned}
\tag{13}
$$

Equation (13) is a valid bound for the maximum attainable ratio of source entropy power to mean square error between the output and input of a system containing a channel perturbed by additive white gaussian noise. Equation (13) is plotted in Figure 1 with n as a parameter.

* Numerous attempts at deriving an expression for the maximum attainable signal-to-noise ratio at the output of a communications system are recorded in the literature, References 3, 4, 5.

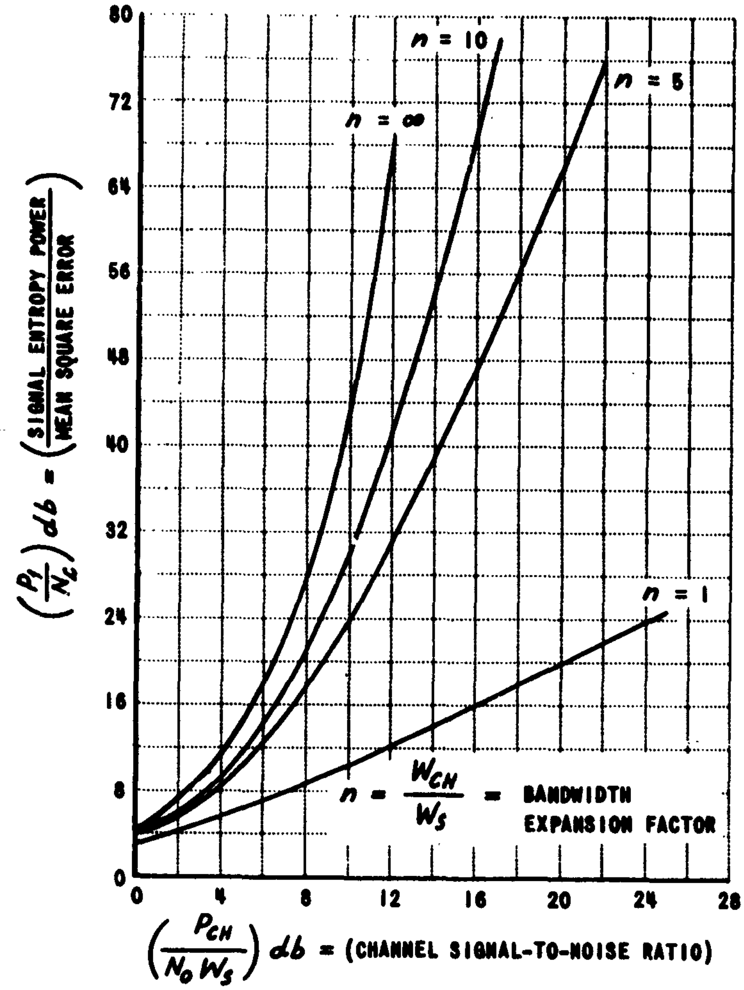


Figure 1 BOUNDS ON THE MAXIMUM ATTAINABLE $\frac{\text{SIGNAL ENTROPY POWER}}{\text{MEAN SQUARE ERROR}}$

In connection with Figure 1, it is again emphasized that signal power and signal entropy power are equal only if the signal has the statistics of white gaussian noise.

In order to gain a better understanding regarding the relationship of the above bounds to the performance of practical communications systems, let us re-examine the manner in which the bounds were derived. The central idea used in the derivation was that of coding which would involve a delay at both the transmitter and the receiving terminals. At the transmitter, an entire sample of duration T is obtained from the random source and, then,

the closest of the $N(T)$ sample functions is selected. A code word representing this sample function is then transmitted*, perturbed by noise, decoded, and the waveform corresponding to the code word reproduced at the receiver. If the channel signal-to-noise ratio is improved and the same code used, the mean square error of the output remains unchanged. (Actually, only the probability of error, which is already assumed arbitrarily small, decreases.) These characteristics are in sharp contrast with those of communications systems using modulators and demodulators which have essentially zero delay.

---

* Note that the characteristics of the coded message need not be simply related to the original waveform; they are, in fact, determined by the characteristics of the channel.

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V

TRANSMISSION OF ANALOG INFORMATION OVER A DIGITAL CHANNEL

INTRODUCTION

The transmission of analog data by means of a digital data link is a practical technique whereby a desired output signal-to-noise ratio can be obtained with reduction of transmitted power but at the expense of increased channel bandwidth. Furthermore, the digital system is adaptable to a variety of digital coding schemes which have been developed for purposes of security or antijam protection. In this chapter some of the characteristics and limitations of such systems will be investigated and compared with recent work by D. Slepian[1,2] which establishes bounds on the error rate performance of the digital link.

Commonly, analog transmission systems are compared on the basis of the output signal-to-noise ratio attainable with a specified channel signal-to-noise ratio, while digital systems are generally analyzed in terms of the probability of error as a function of the channel signal-to-noise ratio. In comparing discrete and continuous systems, or in evaluating the performance of analog-digital-analog systems, it is desirable to establish a relationship between an equivalent analog signal-to-noise ratio and the corresponding error (quantization error and errors due to noise) of the digital channel.

In Chapter IV, it was pointed out that, in general, it is not possible to apply the theorems of information theory in order to establish performance bounds of analog communications systems in terms of the channel and output signal-to-noise ratios. However, when a specific digital system is employed for the transmission of analog data, such relationships may be developed which are useful in comparing the performance of the various systems.

First, it is necessary to establish a reasonable definition of "signal-to-noise ratio". Although we have not been able to obtain a universally applicable definition, the one adopted below is reasonable for the systems under consideration and is also consistent with the signal-to-noise ratio properties that one normally would require for the linear system shown in Figure 1.

In this linear system in which independent noise is added to the input signal, one certainly expects the signal-to-noise ratio at both the input and output of the linear amplifier of gain K to be S/N, where $S = \langle x^2 \rangle$ is the input signal power, and $N = \langle n^2 \rangle$ is the additive noise power. Note that by requiring the output signal-to-noise ratio to be independent of the linear gain, we rule out defining S/N as the ratio

$$\frac{\text{signal power of input signal}}{\text{mean square difference between output and input}} .$$

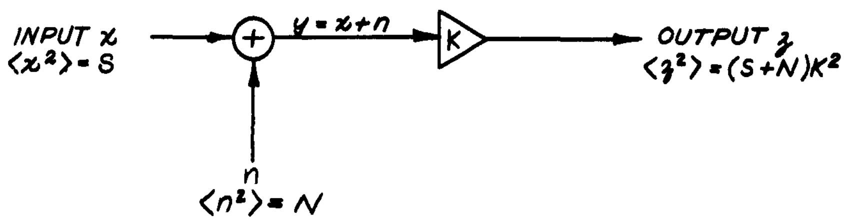


FIGURE 1

In the linear case, we could get the desired result by taking the ratio of

$$\frac{\text{output power due to signal only}}{\text{output power due to noise only}}.$$

However, such a definition fails when there are nonlinear devices in the system and also gives unreasonable results when a linear filter is interposed into the system. We have chosen to define the output S/N as

$$\frac{\text{portion of output correlated with input}}{\text{portion of output uncorrelated with input}}$$

which appears to have a greater range of applicability and, also, gives the desired result for the system shown in Figure 1. Thus,

$$\frac{S}{N} = \frac{\rho^2}{1-\rho^2} \tag{1}$$

where the correlation coefficient ρ is defined by

$$\rho^2 = \frac{\langle xz\rangle^2}{\langle x^2\rangle\langle z^2\rangle} \tag{2}$$

Now, consider the system shown in Figure 2.

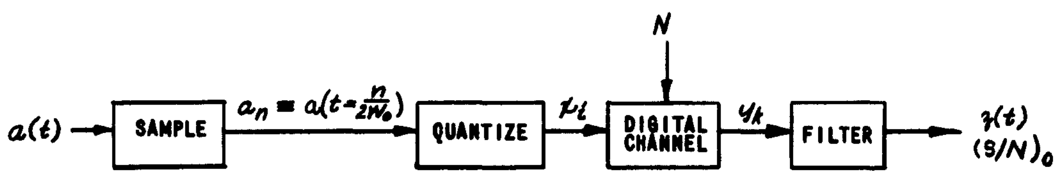


FIGURE 2

Let $a(t)$ be a random function of time which has zero mean and is limited to a bandwidth W_0. By the sampling theorem, $a(t)$ is completely determined by samples a_n taken once every $\frac{1}{2W_0}$ seconds. Let the samples be independent and uniformly distributed over the range $\pm\frac{MA}{2}$. From each sample, one of M quantized samples, x_i, is generated as follows: If $x_i - \frac{A}{2} \leq a_n < x_i + \frac{A}{2}$, then

$$x_i = A\left(i - \frac{M-1}{2}\right); \quad i = 0, 1, 2 \text{---}, M-1 \tag{3}$$

Each x_i is transmitted over a digital channel and received as y_k, with a probability $p(y_k / x_i)$. The sample y_k is given by

$$y_k = A\left(k - \frac{M-1}{2}\right); \quad k = 0, 1, 2, \text{---}, M-1 \tag{4}$$

Since $a(t)$ is sampled at intervals of $\frac{1}{2W_0}$ seconds, the samples y_k occur at this same rate. They are passed through an ideal, unity gain low-pass filter with bandwidth W_0 cps, and the system output $z(t)$ is formed.* Note that the analog input to the system is $a(t)$ and the analog output is $z(t)$; however, they are completely determined by the samples $a_n \equiv a(t = \frac{n}{2W_0})$ and $y_k(t = \frac{n}{2W_0}) = z(t = \frac{n}{2W_0}) \equiv z_n$, respectively. Figure 3 shows the relationship between, and the range of the variables in the system of Figure 2.

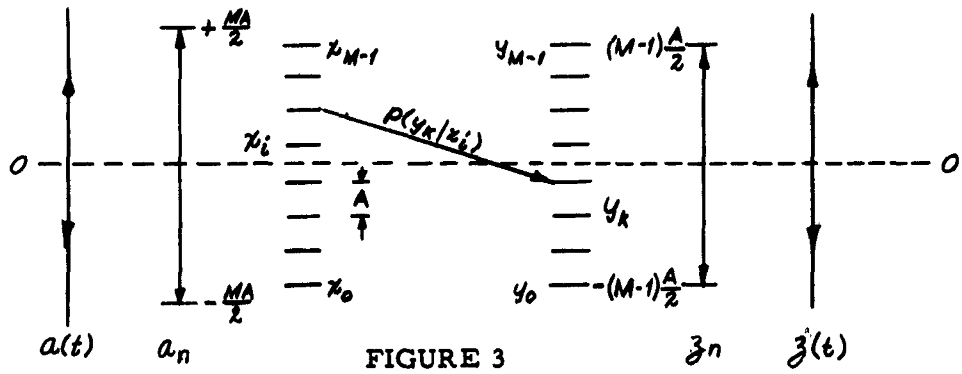


FIGURE 3

The performance of the system for transmitting analog information, described above, will now be obtained in terms of the error rate performance of the digital channel. First, a conventional binary PCM system using an

---

* $z(t) = \sum_{-\infty}^{\infty} z_n \frac{\sin \pi (2W_0 t - n)}{\pi (2W_0 t - n)} = \sum_{-\infty}^{\infty} z_n \operatorname{sinc} t_n$

optimum bipolar binary keying technique will be investigated. Then the performance of a similar binary PCM system will be evaluated under the assumption that the transmitted digits are scrambled so as to produce a uniform distribution of the errors among the incorrect levels. Finally, these results will be compared with performance curves developed from the recent work of Slepian which established performance bounds (probability of error) on digital systems operating over a noisy channel.

Since we are concerned with an analog communication system, the system performance will be described by the relationship between the output signal-to-noise ratio $(S/N)_o$ and the channel signal-to-noise power ratio.

Then, from Equation (1)

$$\left(\frac{S}{N}\right)_o = \frac{\rho}{1-\rho} \tag{5}$$

where

$$\rho^2 = \frac{\langle a z \rangle^2}{\langle a^2 \rangle \langle z^2 \rangle} = \frac{\langle a_n z_n \rangle^2}{\langle a_n^2 \rangle \langle z_n^2 \rangle}^{*} \tag{7}$$

or in terms of $y_k(t = \frac{n}{2W_o}) = z_n = y_k$

$$\rho^2 = \frac{\langle a_n y_k \rangle^2}{\langle a_n^2 \rangle \langle y_k^2 \rangle} \tag{7}$$

THE PCM CHANNEL

Consider a PCM channel where an n -bit number represents each of $M = 2^n$ quantized levels to be transmitted. Since a sample is formed every $\frac{1}{2W_o}$ seconds, the number of bits generated per second is $2W_o n$. Assuming bipolar keying, each bit is received with a probability of error per bit, q , given by[3]

$$q = \frac{1}{2}\left(1 - \operatorname{erf}\sqrt{\frac{E}{N_o}}\right) \tag{8}$$

where E is the energy per bit and N_o is the noise power density. Since the transmitted power S is equal to $2W_o n E$,

$$q = \frac{1}{2}\left(1 - \operatorname{erf}\sqrt{\frac{S}{2W_o n N_o}}\right) = \frac{1}{2}\left[1 - \operatorname{erf}\sqrt{\frac{1}{2n}\left(\frac{S}{N}\right)_i}\,\right] \tag{9}$$

where $(S/N)_i$ is the channel signal-to-noise ratio with the noise referred to

---

*Since $a(t) = \sum_{-\infty}^{\infty} a_n \operatorname{sinc} t_n$ and $\int_{-\infty}^{\infty} \operatorname{sinc} t_j \operatorname{sinc} t_k \, dt = \begin{cases} 1 ; & j = k \\ 0 ; & j \neq k \end{cases}$

a bandwidth W_0 . In bipolar keying, two bits may be transmitted per cycle of output bandwidth; therefore, we obtain $n = W/W_0$, the bandwidth expansion factor.

We now proceed to find ρ in terms of q , the probability of error per bit.

$$\langle a_n^2 \rangle = \frac{1}{MA} \int_{-\frac{MA}{2}}^{+\frac{MA}{2}} a_n^2 \, da_n = \frac{1}{MA} \frac{a_n^3}{3} \Big|_{-\frac{MA}{2}}^{+\frac{MA}{2}} = \frac{M^2 A^2}{12} \tag{10}$$

Since a_n is uniformly distributed, $p(x_i) = \frac{1}{M}$, therefore

$$\langle x_i^2 \rangle = \frac{1}{M} \sum_{i=0}^{M-1} x_i^2 = \frac{A^2}{M} \sum_{i=0}^{M-1} \left(i - \frac{M-1}{2} \right)^2 = \frac{A^2}{M} \sum_{i=0}^{M-1} \left[i^2 - (M-1)i + \frac{(M-1)^2}{4} \right]$$

$$= \frac{A^2}{M} \left[\frac{M(M-1)(2M-1)}{6} - \frac{(M-1)^2 M}{2} + \frac{M(M-1)^2}{4} \right]$$

$$= \frac{A^2}{12} (M^2 - 1) \tag{11}$$

$$\langle y_k^2 \rangle = \sum_{i=0}^{M-1} \sum_{k=0}^{M-1} p(y_k | x_i) \, p(x_i) \, y_k^2 = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} p(y_k | x_i) \, y_k^2 \tag{12}$$

Since the bit errors are independent in binary coding

$$p(y_k | x_i) = (1-q)^{r_{ik}} \, q^{(n - r_{ik})} = p(y_i | x_k) \tag{13}$$

where r_{ik} is the number of correct transitions.

Therefore,

$$\langle y_k^2 \rangle = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} p(y_i | x_k) \, y_k^2$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} y_k^2 = \frac{1}{M} \sum_{i=0}^{M-1} x_i^2 = \langle x_i^2 \rangle = \frac{A^2}{12} (M^2 - 1) \tag{14}$$

where use is made of (3), (4) and (11).

To obtain $\langle a_n y_k \rangle$, we first obtain $\langle a_n y_k \rangle_i$ which is the expected value of $a_n y_k$ given that x_i was sent.

$$\langle a_n y_k \rangle_i = \int_{x_i - \frac{A}{2}}^{x_i + \frac{A}{2}} \sum_{k=0}^{M-1} a_n y_k p(a_n, y_k | x_i) \, da_n \tag{15}$$

Since x_i is being sent, $p(y_k | x_i, a_n) = p(y_k | x_i)$ and $p(a_n | x_i, y_k) = p(a_n | x_i)$ therefore

$$\begin{aligned} \langle a_n y_k \rangle_i &= \int_{x_i - \frac{A}{2}}^{x_i + \frac{A}{2}} \sum_{k=0}^{M-1} a_n y_k p(y_k | x_i) p(a_n | x_i) \, da_n \\ &= \left[\sum_{k=0}^{M-1} y_k p(y_k | x_i) \right] \left[\int_{x_i - \frac{A}{2}}^{x_i + \frac{A}{2}} a_n p(a_n | x_i) \, da_n \right] \\ &= \langle y_k \rangle_i x_i \end{aligned} \tag{16}$$

Before summing on x_i to obtain $\langle a_n y_k \rangle$, we must evaluate $\langle y_k \rangle_i$. The binary number i representing x_i may be written as

$$i = \sum_{s=0}^{n-1} d_s 2^s \qquad \text{where } d_s \text{ is either 1 or 0} \tag{17}$$

Using Equation (4), we may write

$$\langle y_k \rangle_i = A \left\{ \langle k \rangle_i - \frac{M-1}{2} \right\} \tag{18}$$

Let $d_{s,k}$ = 0 or 1 and $d_{s,i}$ = 0 or 1 be the sth bit of the n-bit code representing k and i, respectively. Then

$$\begin{aligned} k - i &= \sum_{s=0}^{n-1} \langle d_{s,k} - d_{s,i} \rangle_i 2^s \\ \langle k - i \rangle_i &= \sum_{s=0}^{n-1} \langle d_{s,k} - d_{s,i} \rangle_i 2^s \end{aligned} \tag{19}$$

Since $\langle d_{s,k} - d_{s,i} \rangle = [(1-q) \times 0] + [q \times 1] = q$ given $d_{s,i} = 0$

and $\langle d_{s,k} - d_{s,i} \rangle = [(1-q) \times 0] + [q \times (-1)] = -q$ given $d_{s,i} = 1$

we can write $\langle d_{s,k} - d_{s,i} \rangle_i = (1 - 2d_{s,i})q$ (20)

Therefore $\langle k-i \rangle_i = \sum_{s=0}^{n-1} (1-2d_{s,i}) q 2^s = q \sum_{s=0}^{n-1} 2^s - 2q \sum_{s=0}^{n-1} d_{s,i} 2^s$

$$\langle k-i \rangle_i = q \langle M-1 \rangle - 2q_i$$

But $\langle k-i \rangle_i = \langle k \rangle_i - i$ (21)

Therefore $\langle k \rangle_i = i + q(M-1) - 2qi$ (22)

From (18) and (22), we obtain

$$\langle y_k \rangle_i = A\left\{\langle k \rangle_i - \frac{M-1}{2}\right\} = A\left\{i + q(M-1) - 2qi - \frac{M-1}{2}\right\}$$

$$= A\left\{\left(i - \frac{M-1}{2}\right) - 2q\left(i - \frac{M-1}{2}\right)\right\}$$

$$= (1-2q) A \left(i - \frac{M-1}{2}\right) = x_i(1-2q) \qquad (23)$$

where use is made of Equation (3).

Using (16) and (23)

$$\langle a_n y_k \rangle_i = \langle y_k \rangle_i x_i = x_i^2 (1-2q)$$

Therefore $\langle a_n y_k \rangle = (1-2q) \langle x_i^2 \rangle$ (24)

We may now obtain ρ^2 for PCM by combining (7), (10), (11), (14) and (24)

$$\rho^2 = \frac{\langle a_n y_k \rangle^2}{\langle a_n^2 \rangle \langle y_k^2 \rangle} = \frac{(1-2q)^2 \langle x_i^2 \rangle^2}{\frac{M^2 A^2}{12} \langle x_i^2 \rangle}$$

$$= \frac{12(1-2q)^2}{M^2 A^2} \frac{A^2}{12}(M^2-1) = \frac{M^2-1}{M^2}(1-2q)^2 \tag{25}$$

Using the definition for $(S/N)_o$, we obtain

$$\left(\frac{S}{N}\right)_o = \frac{\rho^2}{1-\rho^2} = \frac{(M^2-1)(1-2q)^2}{M^2-(M^2-1)(1-2q)^2} \tag{26}$$

Using Equation (9) for bipolar keying

$$(1-2q) = 1-1+\operatorname{erf}\sqrt{\frac{1}{2n}\left(\frac{S}{N}\right)_i} = \operatorname{erf}\sqrt{\frac{1}{2n}\left(\frac{S}{N}\right)_i} \tag{27}$$

Therefore, for binary PCM using bipolar keying, we get

$$\left(\frac{S}{N}\right)_o = \frac{(M^2-1)\left[\operatorname{erf}\sqrt{\frac{1}{2n}\left(\frac{S}{N}\right)_i}\right]^2}{M^2-(M^2-1)\left[\operatorname{erf}\sqrt{\frac{1}{2n}\left(\frac{S}{N}\right)_i}\right]^2} \tag{28}$$

Equation (28) is plotted in Figure 4 for n = 5, 7, 10 and 13.

A MODIFIED BINARY PCM SYSTEM

It is of interest to obtain the performance of a binary PCM system where the errors are distributed uniformly*. To do this, we obtain the probability of error in transmitting an n-bit character using PCM with bipolar keying

$$Q_p = 1-(1-q)^n \tag{29}$$

*Equal probability of all errors can be assured by making the assignment of the sampled values x_i, to the transmitted characters, at random (of course, the assignments must be made in unison at the transmitter and the receiver).

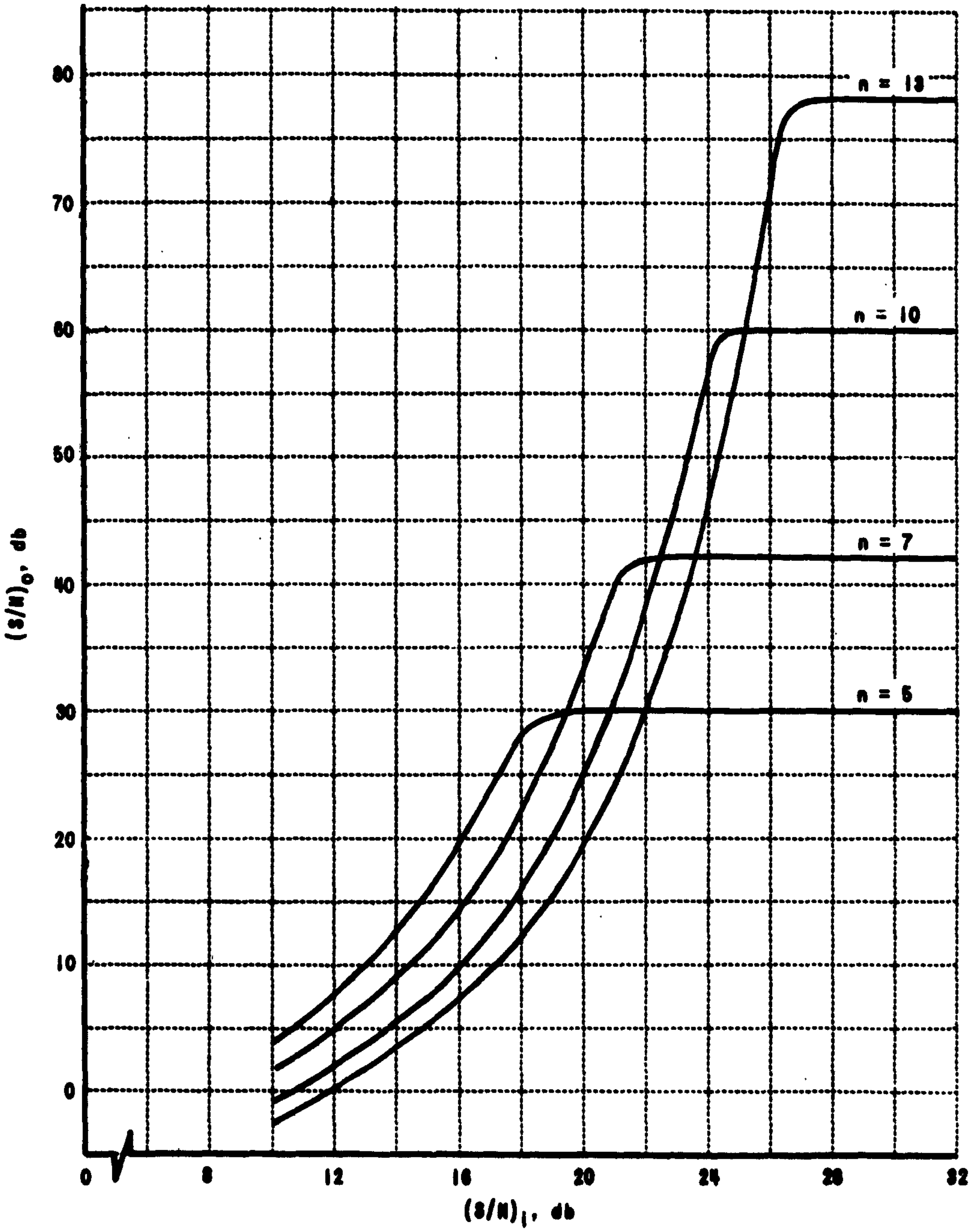


Figure 4 CONVENTIONAL BINARY PCM

Knowing ϱ , as given by Equation (9), we may plot Q_p vs. channel signal-to-noise ratio $\frac{1}{n}(S/N)_i$ (Figure 5). Given the character error Q_p and the assumed distribution, we may write, paralleling the approach used for normal binary PCM,

$$\langle a_n^2 \rangle = \frac{M^2 A^2}{12} \tag{30}$$

$$\langle x_i^2 \rangle = \frac{A^2}{12}(M^2-1) \tag{31}$$

$$\begin{aligned} \langle y_k^2 \rangle &= \frac{1}{M} \sum_{i=0}^{M-1} \sum_{k=0}^{M-1} p(y_k | x_i)\, y_k^2 \\ &= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} p(y_k | x_i)\, y_k^2 \end{aligned} \tag{32}$$

Since, when an error is made, it is assumed to be equally distributed

$$p(y_k | x_i) = p(y_i | x_k) \tag{33}$$

and

$$\langle y_k^2 \rangle = \frac{1}{M} \sum_{k=0}^{M-1} y_k^2 = \langle x_i^2 \rangle = \frac{A^2}{12}(M^2-1) \tag{34}$$

As in conventional PCM from Equation (16)

$$\langle a_n\, y_k \rangle_i = x_i \langle y_k \rangle_i$$

In this case,

$$\begin{aligned} \langle y_k \rangle_i &= (1-Q_p)\, y_i + \frac{Q_p}{M-1} \sum_{\substack{k=0 \\ k \neq i}}^{M-1} y_k \\ &= (1-Q_p)\, y_i - \frac{Q_p}{M-1}\, y_i = y_i \left(1 - Q_p \frac{M}{M-1}\right) \end{aligned} \tag{35}$$

Since

$$y_i = x_i$$

Then

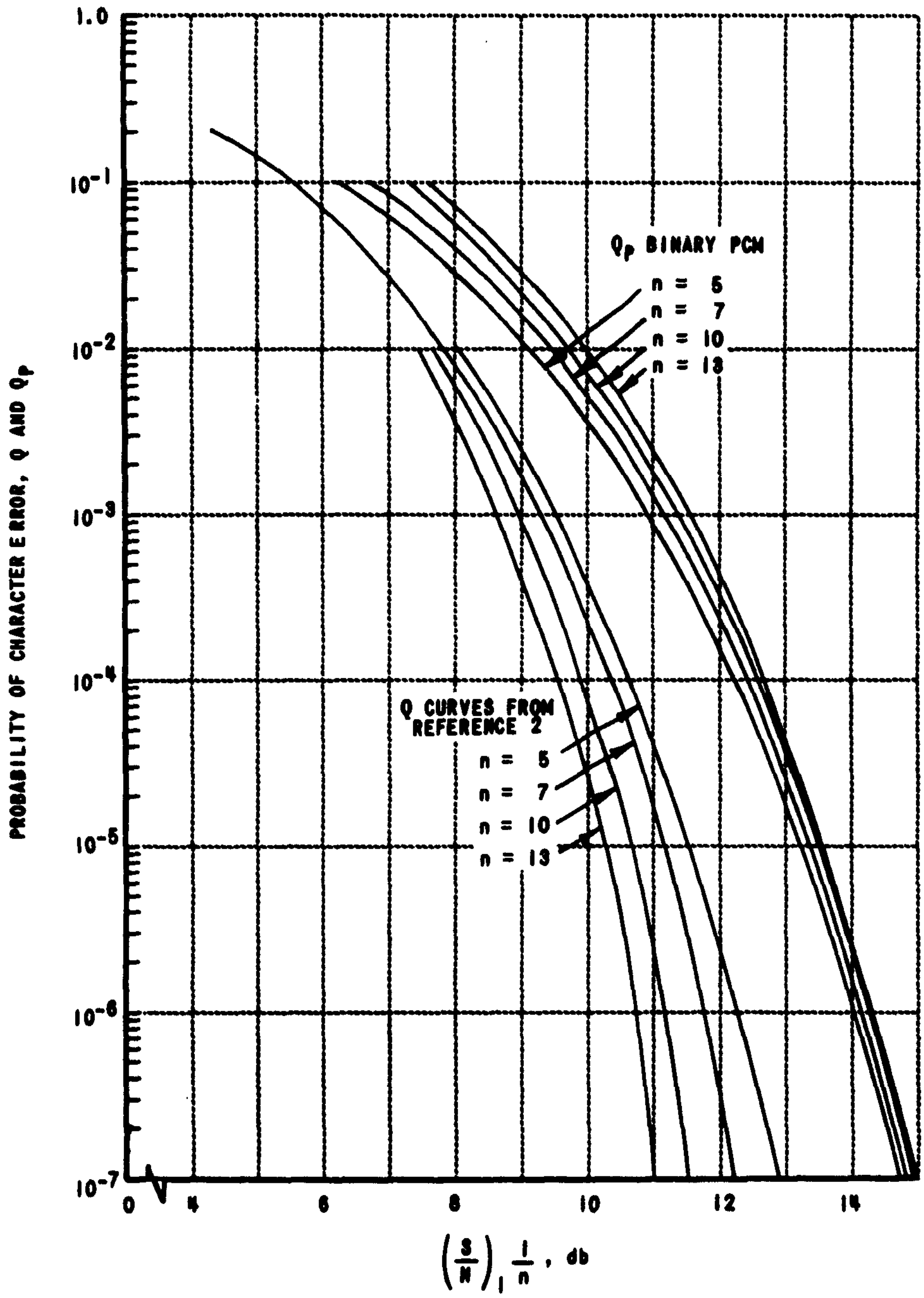
$$\langle a_n\, y_k \rangle_i = x_i^2 \left(1 - Q_p \frac{M}{M-1}\right) \tag{36}$$

and

$$\langle a_n y_k \rangle = \left(1 - Q_p \frac{M}{M-1}\right)^2 \langle x_i^2 \rangle \tag{37}$$

Therefore, from Equations (7), (30), (31), (34) and (37)

$$\rho^2 = \frac{\langle a_n y_k \rangle^2}{\langle x_i^2 \rangle \langle a_n^2 \rangle} = \left(1 - Q_p \frac{M}{M-1}\right)^2 \frac{M^2-1}{M^2} \tag{38}$$


1.0
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8
10
12
14
PROBABILITY OF CHARACTER ERROR, Q AND Q_p
$\left(\frac{S}{N}\right)_1 \frac{1}{n}$, db
Q_p BINARY PCM
n = 5
n = 7
n = 10
n = 13
Q CURVES FROM REFERENCE 2
n = 5
n = 7
n = 10
n = 13

Figure 5

From the definition of $(S/N)_o$, we obtain

$$\left(\frac{S}{N}\right)_o = \frac{(M^2-1)\left(1-Q_p\frac{M}{M-1}\right)^2}{M^2-(M^2-1)\left(1-Q_p\frac{M}{M-1}\right)^2} \tag{39}$$

The probability of character error, Q_p , versus the channel signal-to-noise ratio, $\frac{1}{n}(S/N)_i$ has been computed for binary PCM and is plotted in Figure 5. Using Equation (39) for the modified binary PCM system and the Q_p curves of Figure 5, the output signal-to-noise ratio $(S/N)_o$ has been plotted versus $(S/N)_i$ in Figure 6 for values of n = 5, 7, 10 and 13.

COMPARISON WITH BOUNDS ON DIGITAL SYSTEMS

D. Slepian, in two recent papers[1,2], has applied some of Shannon's results to obtain a bound on the error probability in the transmission of digital data over a noisy channel. Slepian presents curves which give the minimum channel signal-to-noise ratio required to obtain a given probability of error, Q , for various values of n and R/W , where n is equal to $2WT$ (W = channel bandwidth and T is the coding delay) and R/W is the information rate per unit of transmitted bandwidth.

In the case of PCM, the delay time, T , is equal to the sampling period and, hence,

$$\begin{aligned} n &= 2WT \\ &= 2W\frac{1}{2W_o} = \frac{W}{W_o} \end{aligned} \tag{40}$$

Then,

$$\begin{aligned} \frac{R}{W} &= \frac{2W_o}{W}\log_{10} M \\ &= \frac{2}{n}\log_{10} M = \frac{2}{n}\log_{10} 2^n = 2\log_{10} 2 \\ &\cong 0.6 \end{aligned} \tag{41}$$

Since Q is the probability of error in the transmission of a code word T seconds long, it is equal (in the PCM system where $T = \frac{1}{2W_o}$) to the probability of error in the transmission of x_i .

Figure 5, obtained from cross plots of data presented in Reference 2, shows curves for Q versus $\frac{1}{n}(S/N)_i$ for various values of n .

In order to obtain an output signal-to-noise ratio from these curves, we make the assumption that when an error is made, it is uniformly distributed among M-1 levels. (As was previously noted, this assumption may always be satisfied.) This allows us to use the intermediate results obtained in Equation (39) for the modified binary PCM system.

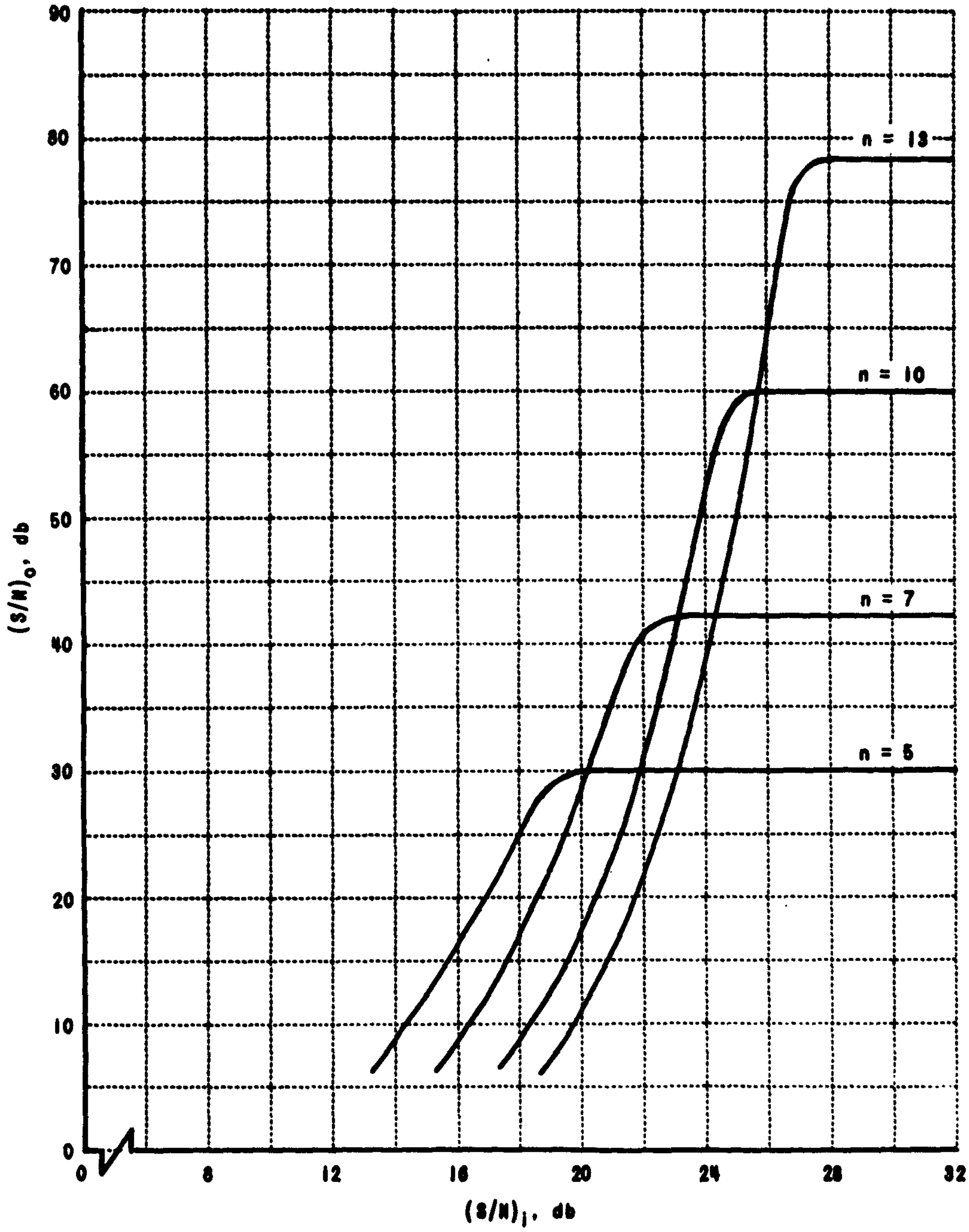


Figure 6 MODIFIED BINARY PCM (WITH UNIFORM ERROR DISTRIBUTION)

Hence,

$$\left(\frac{S}{N}\right)_o = \frac{(M^2-1)\left(1-Q\frac{M}{M-1}\right)^2}{M^2-(M^2-1)\left(1-Q\frac{M}{M-1}\right)^2} \tag{42}$$

Equation (42) relates the output signal-to-noise ratio to the probability of a character error Q. Using the Q curves of Figure 5 and Equation (42), the output signal-to-noise ratio versus channel signal-to-noise ratio $(S/N)_i$ has been plotted in Figure 7.

CALCULATION OF RATIO OF SOURCE ENTROPY POWER TO MSE

In Chapter IV of this report, bounds on the maximum attainable ratio of source entropy power P_1 to mean square error (MSE) have been presented. This ratio may be obtained for the systems under consideration in this chapter and compared with the results shown in Figure 1 of Chapter IV. The entropy power P_1 of the source is defined by

$$P_1 = \frac{1}{2\pi e} e^{2H'} \tag{43}$$

where H' is the entropy per degree of freedom of the source. Since the samples of the source are independent,

$$H' = -\int p(a_n)\left[\ln p(a_n)\right] da_n \tag{44}$$

with

$$p(a_n) = \begin{cases} \frac{1}{MA}, & |a_n| < \frac{MA}{2} \\ 0, & \text{elsewhere} \end{cases} \tag{45}$$

Then

$$H' = -\int_{-\frac{MA}{2}}^{+\frac{MA}{2}} \frac{1}{MA}\left(\ln\frac{1}{MA}\right) da_n = -\ln\frac{1}{MA} \tag{46}$$

and combining (43) and (46),

$$P_1 = \frac{1}{2\pi e} e^{-2\ln\frac{1}{MA}} = \frac{1}{2\pi e} e^{\ln(MA)^2} = \frac{(MA)^2}{2\pi e} \tag{47}$$

Since $\langle a_n^2\rangle = \frac{M^2A^2}{12}$ (from (10))

$$P_1 = \frac{6}{\pi e}\langle a_n^2\rangle = 0.703\langle a_n^2\rangle \tag{48}$$

and

$$\frac{P_1}{MSE} = 0.703\frac{\langle a_n^2\rangle}{MSE} \tag{49}$$

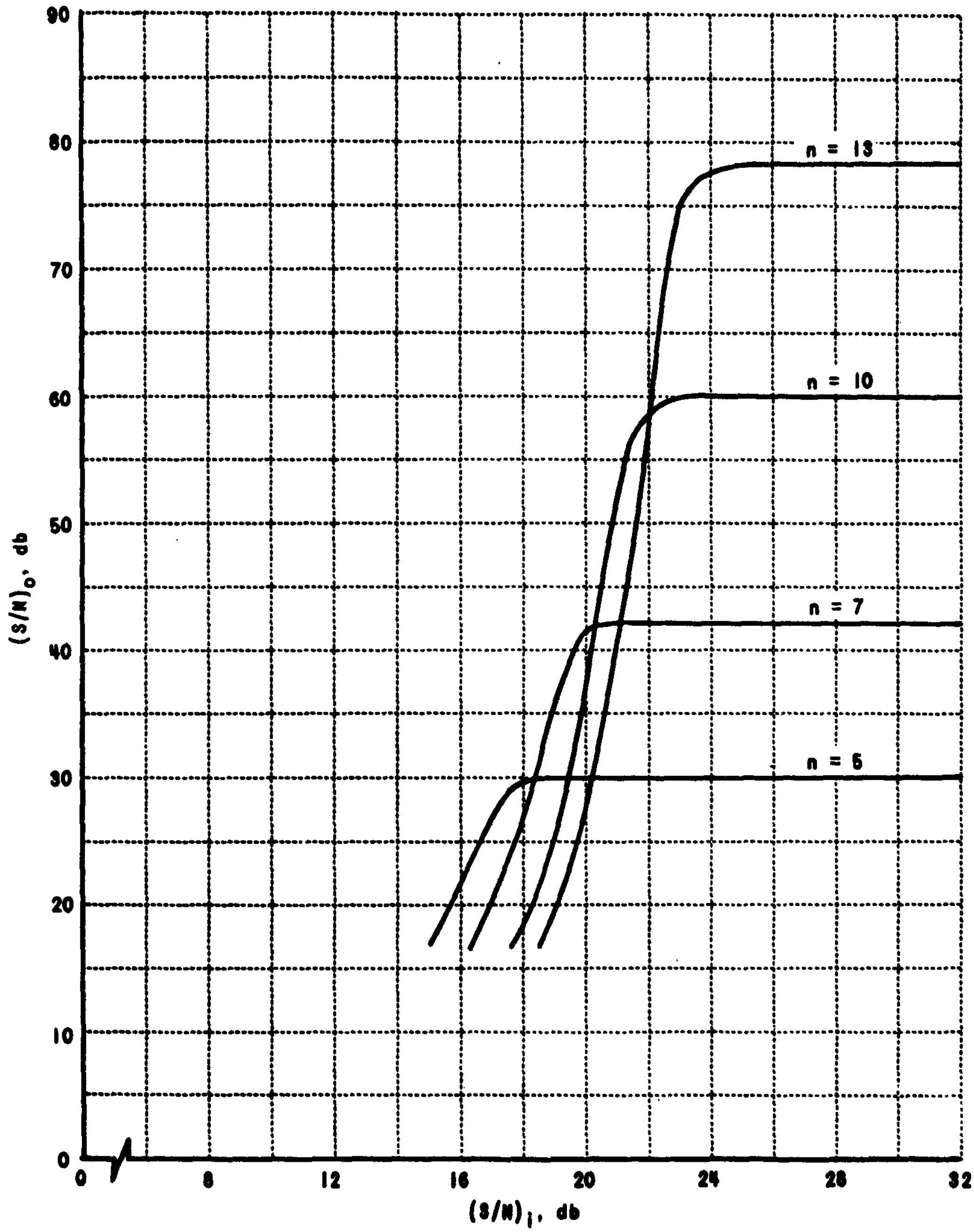


Figure 7 PCM PERFORMANCE BASED ON DIGITAL BOUNDS FROM REFERENCE 2

In order to minimize the MSE, y_k is modified by a gain factor λ.

$$\text{MSE} = \langle (a_n - \lambda y_k)^2 \rangle = \langle a_n^2 \rangle + \lambda^2 \langle y_k^2 \rangle - 2\lambda \langle a_n y_k \rangle \tag{50}$$

From (24) and (37) and since $\langle y_k^2 \rangle = \langle x_i^2 \rangle$

$$\text{MSE} = \langle a_n^2 \rangle + \lambda^2 \langle x_i^2 \rangle - 2\lambda\phi \langle x_i^2 \rangle \tag{51}$$

where $\phi = (1-2q)$ for conventional binary PCM with bit error rate q.

and $\phi = \left(1 - Q_p \frac{M}{M-1}\right)$ for PCM with a uniform character error distribution.

To minimize the MSE with respect to λ,

$$\frac{d}{d\lambda}(MSE) = 2\lambda \langle x_i^2 \rangle - 2\phi \langle x_i^2 \rangle = 0$$

from which $\lambda = \phi$

Therefore, the minimum MSE is given by

$$MSE = \langle a_n^2 \rangle + \phi^2 \langle x_i^2 \rangle - 2\phi^2 \langle x_i^2 \rangle \tag{52}$$

Then

$$\frac{\langle a_n^2 \rangle}{MSE} = \frac{\langle a_n^2 \rangle}{\langle a_n^2 \rangle - \phi^2 \langle x_i^2 \rangle} = \frac{1}{1 - \phi^2 \dfrac{\langle x_i^2 \rangle}{\langle a_n^2 \rangle}}$$

From (10), (11), (30) and (31)

$$\frac{\langle x_i^2 \rangle}{\langle a_n^2 \rangle} = \frac{M^2-1}{M^2}$$

and, therefore,

$$\frac{\langle a_n^2 \rangle}{MSE} = \frac{1}{1 - \dfrac{M^2-1}{M^2}\phi^2} \tag{53}$$

For the case of conventional binary PCM,

$$\frac{P_1}{MSE} = \frac{0.703}{1 - \dfrac{M^2-1}{M^2}(1-2q)^2} \tag{54}$$

and for modified binary PCM (uniformly distributed errors)

$$\frac{P_1}{MSE} = \frac{0.703}{1 - \dfrac{M^2-1}{M^2}\left(1 - Q\dfrac{M}{M-1}\right)^2} \tag{55}$$

and for the PCM system based on the error rate bounds given in Figure 4 (Q curves)

$$\frac{P_1}{MSE} = \frac{0.703}{1 - \frac{M^2-1}{M^2}\left(1 - Q\,\frac{M}{M-1}\right)^2} \qquad (56)$$

Equations (55) and (56) are plotted in Figure 9 for $n = 5$ and $n = 10$ ($M = 32$ and 1024) along with the bounds obtained in Chapter IV.

SUMMARY AND DISCUSSION OF RESULTS

The application of PCM techniques to the transmission of continuous data has been investigated. Equation (28) relates the output signal-to-noise ratio for conventional binary PCM to the number of quantization levels and the channel signal-to-noise ratio. These results are plotted in Figure 4 for various bandwidth-expansion factors.

Figure 6 is a similar plot in which the output signal-to-noise ratio is plotted as a function of the channel signal-to-noise ratio for various values of n when the binary PCM system is modified such that the errors are uniformly distributed among the incorrect levels.

Then, from Equation (42) and using the bounds on the error probability as given by the Q curves in Figure 5, we obtain a bound on the output signal-to-noise ratio as a function of channel signal-to-noise ratio for difference values of n. This represents an upper bound (but not necessarily the lowest upper bound) on the performance of the modified PCM system with a uniform error distribution and these results are plotted in Figure 7.

For purposes of comparing different communication techniques, it is desirable to exhibit the performance characteristics of these systems by a curve representing the envelope of the knees of the curves in Figures 4, 6 and 7. Such curves for the two PCM systems are shown in Figure 8 as curves C and D and give, for a particular desired output signal-to-noise ratio, the minimum channel signal-to-noise ratio required. Similarly, curve B in Figure 8 is the envelope of the knees of the curves given in Figure 7, based on Slepian's work.

In the system shown in Figure 2, the information signal is converted into a discrete signal source of M levels. Since one sample is obtained every $\frac{1}{2W_0}$ seconds, the maximum information rate of this source is given by

$$R_0 = 2W_0 \log_2 M \quad \text{bits/sec.} \qquad (57)$$

Note that R may be less than R_0 if the sample values are correlated or if the M levels are not equally likely. Shannon has shown that it is possible to transmit a message with an arbitrarily small error probability

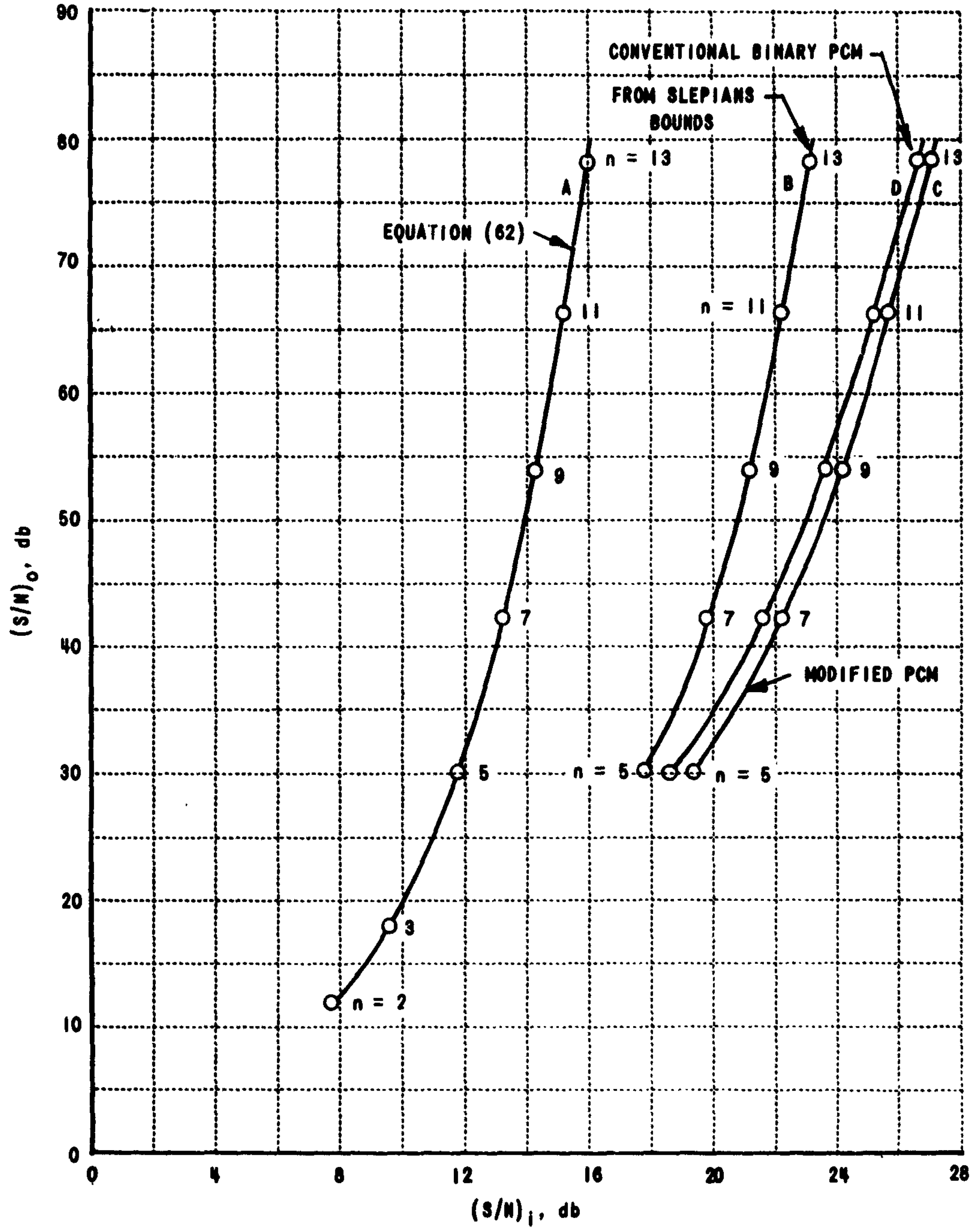

CONVENTIONAL BINARY PCM
FROM SLEPIANS BOUNDS
EQUATION (62)
MODIFIED PCM
A
B
C
D
n = 2
3
5
7
9
11
n = 13
n = 5
n = 11
13
(S/N)o, db
(S/N)i, db
0
10
20
30
40
50
60
70
80
90
0
4
8
12
16
20
24
28

Figure 8

over a channel of bandwidth W perturbed by additive white gaussian noise of average power N and with average message power S, provided

$$R < W \log_2 \left(1+\frac{S}{N}\right). \tag{58}$$

Letting

$$R_o = 2W_o \log_2 M = W \log_2 \left(1+\frac{S}{N}\right) \tag{59}$$

we can find the minimum S/N required to reproduce the M level signal with a probability of error as small as desired. With reference to the system of Figure 2, a zero error probability will result in an output $(S/N)_o = \rho^2/(1-\rho^2)$ given by Equation (26)

$$(S/N)_o = M^2 - 1 \tag{60}$$

Solving for M and substituting in Equation (59) we obtain

$$W_o \log_2 \left[1+\left(\frac{S}{N}\right)_o\right] = W \log_2 \left[1+\left(\frac{S}{N}\right)_i \frac{1}{n}\right] . \tag{61}$$

Solving for $(S/N)_o$

$$\left(\frac{S}{N}\right)_o = \left[1+\left(\frac{S}{N}\right)_i \frac{1}{n}\right]^n - 1 \tag{62}$$

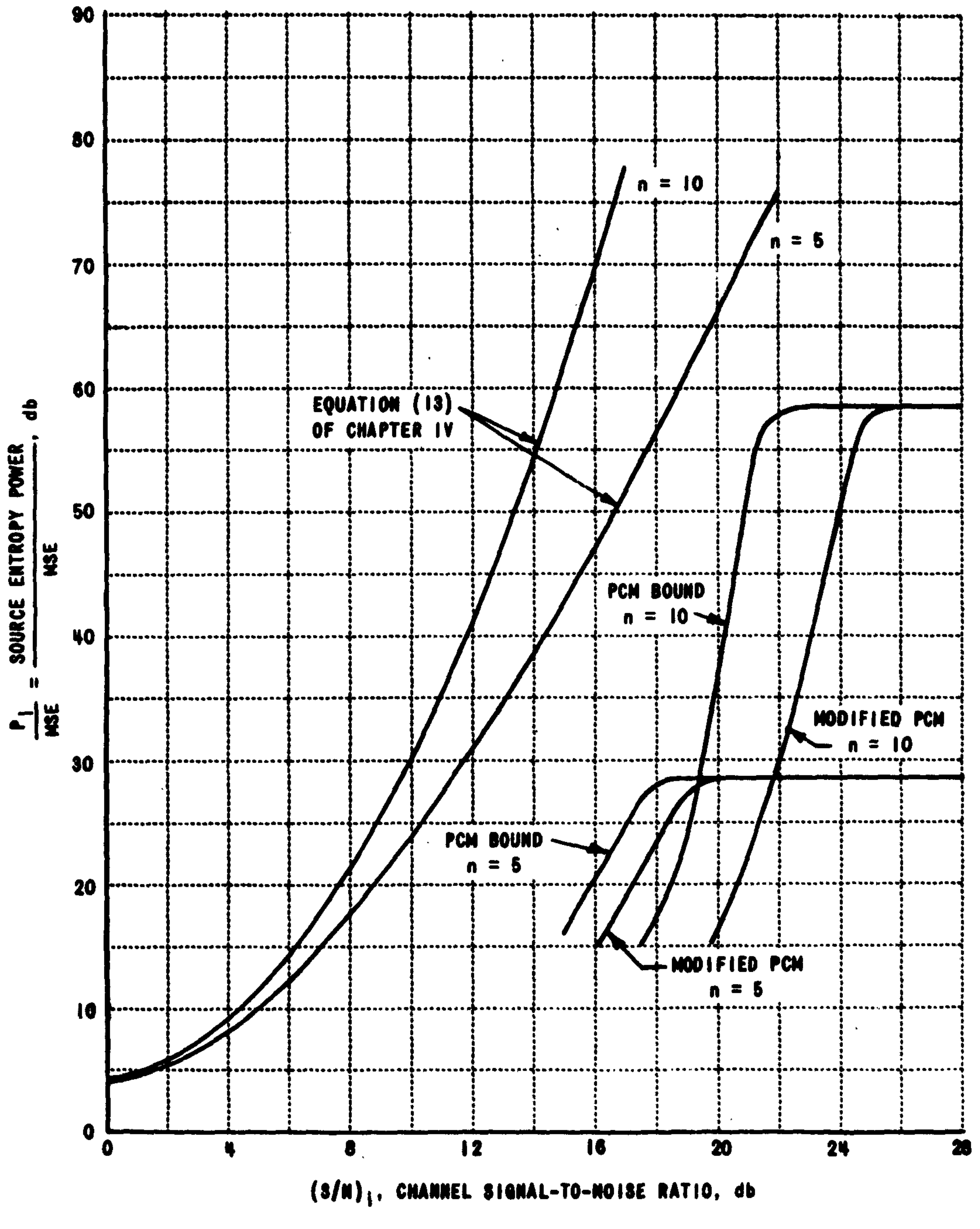
Equation (62) is also plotted (Curve A) in Figure 8 and relates, for a desired $(S/N)_o$ (and a given bandwidth expansion factor), the minimum value of $(S/N)_i$ required. It should be emphasized that this minimum $(S/N)_i$ is achieved only with a sufficiently long and complex encoding process which entails a delay approaching infinity while the other curves represent systems in which the delay is equal to $\frac{1}{2W_o}$.

Curve B represents a bound on the performance of a system with a uniform error distribution and having the same number of degrees of freedom as a binary PCM system (R/W = 0.6) and, therefore, may be compared to the modified PCM system (Curve D). We note, again, that Curve B is an upper bound but not necessarily the lowest upper bound and, hence, does not indicate that a system exists which can do as well. However, we see that only a few db of $(S/N)_i$ separate this upper bound from the modified PCM system represented by Curve D. The output signal-to-noise ratio in these PCM systems is found to be a function of the manner in which the errors are distributed among the incorrect levels, as is demonstrated in Figures 4 and 6. It is observed that the difference in the error distribution has its greatest effect at very low signal-to-noise ratios and has relatively less effect on the position of the knees of the curves. As would be expected, for the PCM systems considered, the signal-to-noise ratio performance is degraded by imposing the condition of uniform error

distribution as compared to the conventional binary PCM system where the error distribution is more favorable.

The performance bounds developed by Slepain in Reference 2 allow one to establish bounds on the performance of various systems considerably more complex than the binary PCM systems considered here. One might consider coding schemes in which a coding and a decoding delay, T, is accepted (where $T > \frac{1}{2W_o}$) but where the information rate to channel bandwidth ratio remains unchanged. Thus, $n = 2WT$ has been increased and, from Reference 2, we can establish bounds on the performance of these systems. It should be noted that Reference 2 provides not only a performance bound that cannot be exceeded, but also provides curves which define a performance level that is, at least, obtainable. It should also be noted that when a code group is decoded, not all of the data words will necessarily be incorrect. The distribution of these data word errors is not specified in Reference 2 and would have to be known from the characteristics of the given coding system in order to compute the system output signal-to-noise ratio. However, if the resulting error probability, Q, is sufficiently small, the only contribution to the output noise will be due to quantization noise, and the limiting output signal-to-noise ratio may be obtained from the expression given by Equation (62).

In Chapter IV, bounds on the maximum attainable ratio of signal entropy power to mean square error were derived and plotted in Figure 1 of that chapter. This ratio also has been computed for the PCM systems considered in this chapter, and some of these results are plotted in Figure 9 along with the results from the previous chapter. The P_1/MSE for the modified PCM (uniform error distribution) is plotted from Equation (55) for $n = 5$ and $n = 10$, and the P_1/MSE for the PCM system based on the probability of error bounds of Slepian are also plotted for the same values of n. These curves are very similar to the $(S/N)_o$ curves plotted in Figures 6 and 7 and show the threshold of the modified PCM system to fall a few db from the threshold of the curves developed from Slepian's bounds. However, we see that the bounds on the P_1/MSE obtained from Chapter IV fall several db to the left of the PCM curves, which reflects the relatively poor utilization of the theoretical channel capacity by a binary signaling system for this range of channel signal-to-noise ratios. It may be further noted that the bounds obtained in Chapter IV apply to all signaling schemes whereas Slepian's bounds apply only to equal energy signaling methods and that binary PCM is in the latter class.



90
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30
20
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4
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12
16
20
24
28
n = 10
n = 5
EQUATION (13)
OF CHAPTER IV
PCM BOUND
n = 10
MODIFIED PCM
n = 10
PCM BOUND
n = 5
MODIFIED PCM
n = 5
$\frac{P_1}{MSE}$ = $\frac{\text{SOURCE ENTROPY POWER}}{\text{MSE}}$, db
$(S/N)_i$, CHANNEL SIGNAL-TO-NOISE RATIO, db

Figure 9

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3. Becker, H. and Lawton, J., <u>Theoretical Comparison of Binary Data Transmission Systems</u>, CAL Report No. CA-1172-S-1 RADC TR 58-91, May 1958.

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VI

APPLICATION OF A PARTIAL ORDERING OF CHANNELS TO THE COMPARISON OF DIGITAL DATA SYSTEMS

The problem of comparing digital systems with different size transmission alphabets has recently been considered by Wolf.[1] The method of comparing N-ary systems described by Wolf is as follows. A K-ary stream of information digits is converted to an N-ary stream of transmission digits and the received N-ary transmission digits are then converted back to a stream of K-ary digits. The probability of error per K-ary character in the output stream is denoted by $P_K^{(N)}$. In comparing two systems having transmission alphabets of size N_1 and N_2, if $P_K^{(N_1)} < P_K^{(N_2)}$, the N_1-ary system is more reliable than the N_2-ary system for transmitting K-ary information. A reversal of the inequality reverses the ordering of the reliabilities of the systems.

Wolf illustrates the surprising result that the relative performance of the systems for fixed N_1 and N_2 may depend upon the size K of the information alphabet for which the error probabilities are computed and then compared. Thus, it is possible that $P_{K_1}^{(N_1)} < P_{K_1}^{(N_2)}$ for a comparison on the basis of K_1-ary information digits while for K_2-ary digits $(K_2 \neq K_1)$ $P_{K_2}^{(N_1)} > P_{K_2}^{(N_2)}$. Now, for optimum coherent detection of N orthogonal signals of equal energy E, chosen at the transmitter with equal probability, and corrupted by additive white gaussian noise with zero mean and spectral density N_0 w/cps, the probability of error per N-ary character is[2]

$$P_N = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\chi - \sqrt{\frac{2E}{N_0}}\right)^2\right] \phi^{N-1}(\chi)\, d\chi \tag{1}$$

where

$$\phi(\chi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\chi} \exp\left(-\frac{1}{2} y^2\right) dy \tag{2}$$

If $K = 2$ and N is an integral power of 2, then[3]

$$P_2^{(N)} = \frac{N}{2(N-1)} P_N \tag{3}$$

By reference to Figure 1, one sees that $P_2^{(2)} < P_2^{(32)}$ for low signal-to-noise ratios, that is, when the energy-per-information-bit/noise-power-density is less than approximately -5db. However, Wolf also shows that in this same range $P_{32}^{(2)} > P_{32}^{(32)}$, which illustrates the dependence of this method of comparison upon the size of the K-ary comparison alphabet. A dependence upon signal-to-noise ratio is also evident from Figure 1 in that $P_2^{(2)} > P_2^{(32)}$ above approximately -5 db.

Thus far, only digital systems characterized by square, symmetric transition probability matrices between input and output symbols have been discussed. We will now describe a basis for comparing arbitrary discrete communication channels, i. e., systems characterized by general rectangular probability matrices, which is both intuitively satisfying and contains within it the results described above. This comparison method is based upon the partial ordering of communication channels which was introduced by Shannon[4] and extended in RADC-TDR-62-134. The class of all discrete memoryless channels is partially ordered with respect to a relation of inclusion, written $\supseteq$, i. e., if K_1, K_2 and K_3 are any channels,

$$\begin{aligned} &\text{(i)} \quad K_1 \supseteq K_1 \\ &\text{(ii) If} \quad K_1 \supseteq K_2 \quad \text{and} \quad K_2 \supseteq K_1, \text{ then } K_1 = K_2 \\ &\text{(iii) If} \quad K_1 \supseteq K_2 \quad \text{and} \quad K_2 \supseteq K_3, \text{ then } K_1 \supseteq K_3 \end{aligned} \tag{4}$$

The inclusion relation itself can be defined in terms of the transition probability matrices which pertain between channel input and output symbols. Keeping in mind that the transition probability matrix of a cascade of channels is the matrix product of the individual channel transition probability matrices, given a channel represented by a transition probability matrix W one can, by employing pre- and post-channel pairs with matrices R_α and T_α with probability g_α, obtain a channel represented by the matrix Q where

$$Q = \sum_\alpha g_\alpha R_\alpha W T_\alpha \tag{5}$$

The channel characterized by W includes that characterized by Q, written $W \supseteq Q$, if Equation (5) is true for some set of R_α, T_α and g_α. Thus, in words, W includes Q if W can be made to behave as Q.

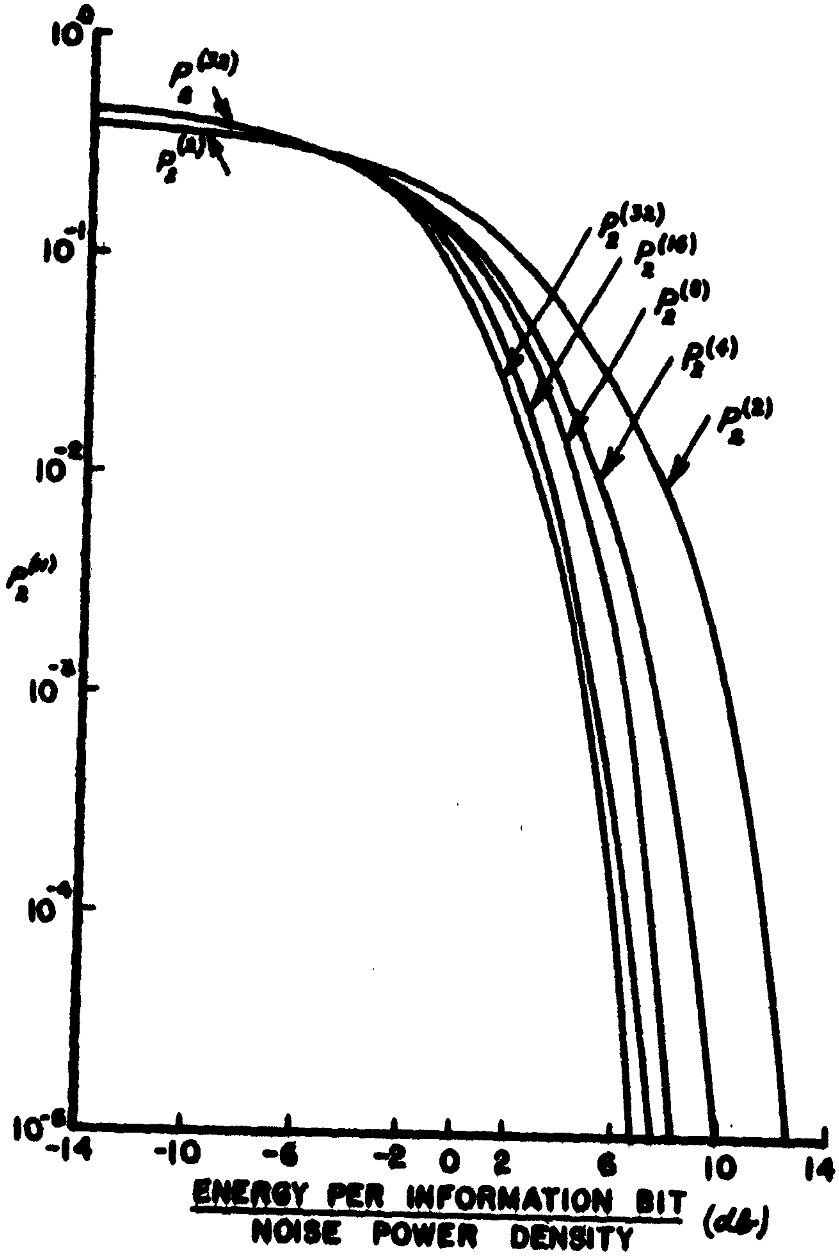

Figure I PROBABILITY OF ERROR IN BINARY INFORMATION DIGITS FOR N ORTHOGONAL TRANSMISSION SIGNALS (FROM WOLF[4])

If $W \supseteq Q$ and $Q \supseteq W$, W and Q are said to be equivalent, written $W \equiv Q$. This is a mathematical equivalence relation and, as such, partitions the class of all transition probability matrices into disjoint equivalence classes. Channels are identified with equivalence classes of stochastic matrices, which accounts for property (ii) of (4).

Given any two channels K_1 and K_2, if $K_1 \supseteq K_2$, it is reasonable to say that K_1 is at least as reliable as K_2 since K_1 can at least duplicate the performance of K_2. However, this in itself does not completely resolve the question of comparing two channels, since it is possible that one has neither $K_1 \supseteq K_2$ nor $K_2 \supseteq K_1$ for a given pair of channels, i.e., neither channel includes the other. It will be seen that this is exactly the case for the binary and 32-ary channels considered earlier for an energy-per-information-bit/noise-power-density less than about -5 db, whereas for values greater than this the 32-ary channel includes the binary channel.

At this stage the channels being considered are quite general, even to the extent of having different size input and output alphabets. As a special case, consider the symmetric channels, which are defined as follows. A channel is called symmetric if for some K and some p the equivalence class of transition probability matrices constituting the channel contains the matrix $P = [p_{ij}]$ where

$$\begin{aligned} p_{ij} &= p, & i &= j \\ &= \frac{1-p}{K-1} & i &\neq j \\ i, j &= 1, 2, \cdots, K. \end{aligned} \tag{6}$$

Such a channel is completely specified by K and p and can be denoted $C_{K,p}$. The method described here for comparing arbitrary channels is as follows.

Given arbitrary channels K_1 and K_2 let p_1 be the maximum p for which $K_1 \supseteq C_{K,p}$ and p_2 the maximum p for which $K_2 \supseteq C_{K,p}$. If $p_1 > p_2$, then K_1 is more reliable than K_2 for transmitting K-ary information. In terms of error probabilities this condition is $1-p_1 < 1-p_2$. Thus, K_1 is more reliable than K_2 for transmitting K-ary information if K_1 can be made to behave as a K-ary symmetric channel with a smaller probability of error than is the case for K_2.

An immediate consequence of the above is that if K_1 includes K_2, then K_1 is at least as reliable as K_2 for transmitting K-ary information for all K. This follows from Property (iii) of (4).

Consider now the comparison of N-ary systems of the type described earlier, with error probabilities given by Equation (1). These are symmetric channels, and their comparison then centers about inclusion relationships between symmetric channels. The following result, which was first derived by Walbesser[5], is concerned with the structure of Shannon's partial ordering of symmetric channels.

Theorem: A necessary and sufficient condition that $C_{N,p} \supseteq C_{R,t}$, $N, R > 1$ is that t lie in the closed interval:

$$\left.\begin{array}{ll}
\text{I.} & R \geq N \\
& \dfrac{1-p}{N-1}\dfrac{N}{R} \leq t \leq \dfrac{N}{R}p, \qquad p \geq \dfrac{1}{N} \\
& \dfrac{N}{R}p \leq t \leq \dfrac{1-p}{N-1}\dfrac{N}{R}, \qquad p \leq \dfrac{1}{N} \\
\text{II.} & R \leq N \\
& \dfrac{1-p}{N-1}\dfrac{N}{R} \leq t \leq \dfrac{R-1}{N-1}\dfrac{N}{R}p + \dfrac{N-R}{R(N-1)}, \quad p \geq \dfrac{1}{N} \\
& \dfrac{R-1}{N-1}\dfrac{N}{R}p + \dfrac{N-R}{R(N-1)} \leq t \leq \dfrac{1-p}{N-1}\dfrac{N}{R}, \quad p \leq \dfrac{1}{N}
\end{array}\right\} \tag{7}$$

(A proof of this theorem is given in the Appendix.) This is a minor extension of the results presented in RADC-TDR-63-134 in that symmetric channels $C_{N,p}$ for which $p < \frac{1}{N}$ are also considered. The inclusion relations amongst the symmetric channels, as determined by the above conditions, are illustrated graphically in Figure 2. In the unshaded regions neither channel includes the other. In what follows, we limit ourselves to the more realistic N-ary symmetric channels for which $p > \frac{1}{N}$

Let us now compare an N_1-ary and N_2-ary channel with respect to their ability to transmit K-ary information. Denote the channels by C_{N_1,p_1} and C_{N_2,p_2}. For $K \geq N_1, N_2$, it is readily determined from (7) that C_{N_1,p_1} is more reliable than C_{N_2,p_2} if

$$N_2 p_2 < N_1 p_1 \tag{8}$$

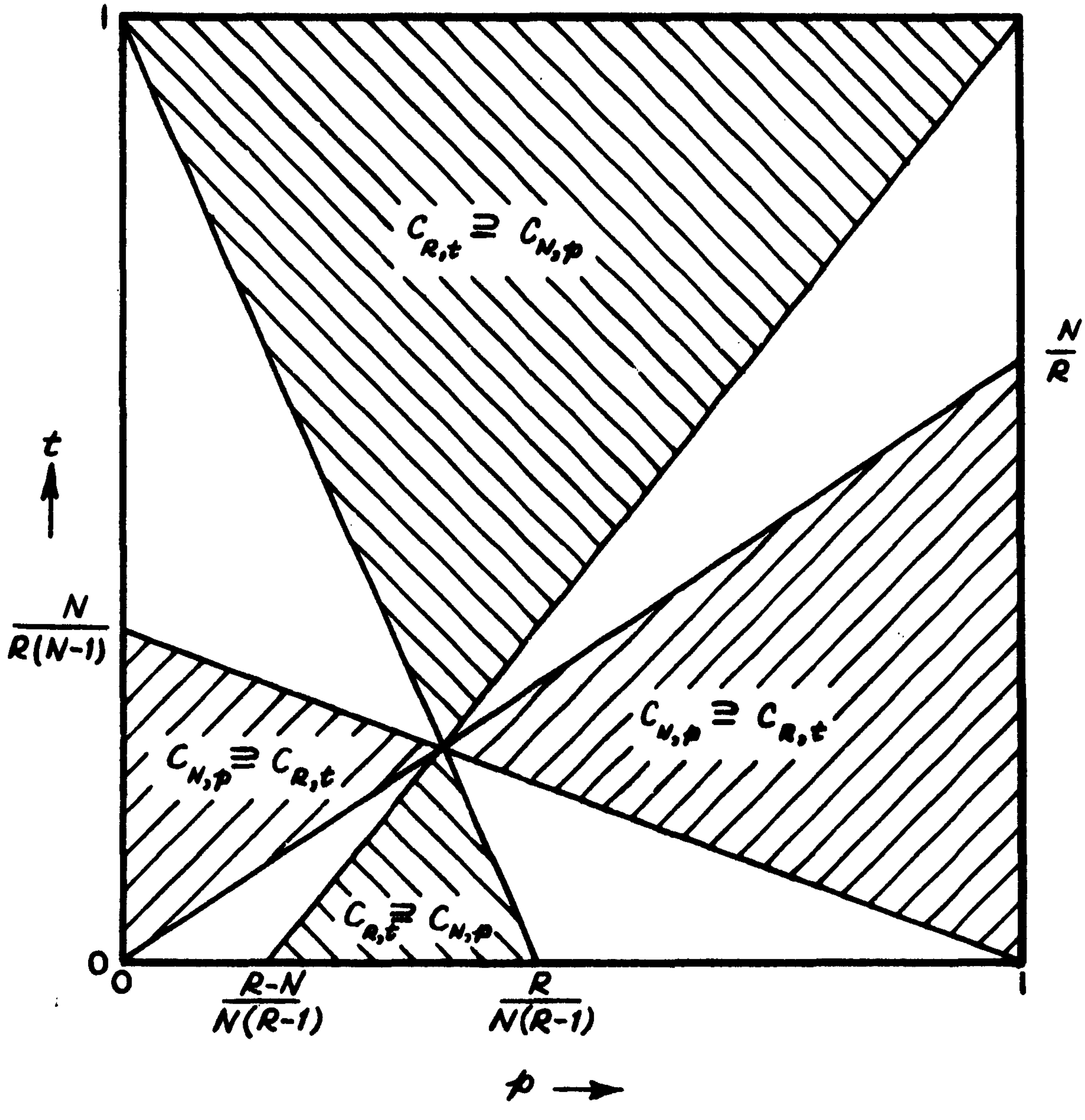

Figure 2 INCLUSION RELATIONSHIPS AMONGST THE SYMMETRIC CHANNELS ($R \geq N$)

In terms of the error probabilities $P_{N_1} = 1-p_1$, $P_{N_2} = 1-p_2$, (8) takes the form

$$P_{N_1} < 1 - \frac{N_2}{N_1} + \frac{N_2}{N_1} P_{N_2} \tag{9}$$

Note that this result is independent of K as long as K is greater than or equal to both N_1 and N_2 . In particular, for N_1 = 32 and N_2 = 2, using the numerical results from Figure 1 together with Equation (3), it is found that

$$P_{32} < 1 - \frac{1}{16} + \frac{1}{16} P_2$$

Thus, the 32-ary channel is more reliable than the binary channel for transmitting K-ary information for all $K \geq$ 32. Figure 3 depicts the situation for the case of K = 32. The curves of P_2 and P_{32} give the performance of the systems to be compared. For a given energy-per-information-bit/noise-power-density, the 32-ary channel includes all 32-ary channels with error probability lying in the dashed region, whereas the binary channel includes those 32-ary channels lying in the shaded region. It is seen that the given 32-ary channel includes all 32-ary channels included by the given binary channel and more. This is true for all abscissa values shown.

Suppose now that $K \leq N_1, N_2$. It is readily determined from (7) that C_{N_1, p_1} is more reliable than C_{N_2, p_2} if

$$\frac{N_1}{N_1 - 1}\left(1-p_1\right) < \frac{N_2}{N_2 - 1}\left(1-p_2\right) \tag{10}$$

or, in terms of error probabilities,

$$\frac{N_1}{N_1 - 1} P_{N_1} < \frac{N_2}{N_2 - 1} P_{N_2} \tag{11}$$

Again, this result is independent of K as long as K is less than or equal to both N_1 and N_2 . Even more striking is the fact that in the notation of Equation (3) this can be written

$$P_2^{(N_1)} < P_2^{(N_2)} \tag{12}$$

which coincides with Wolf's method of comparison for K =2. The result here, however, requires only that $K \leq N_1, N_2$ and places no other restrictions on N_1 and N_2 .

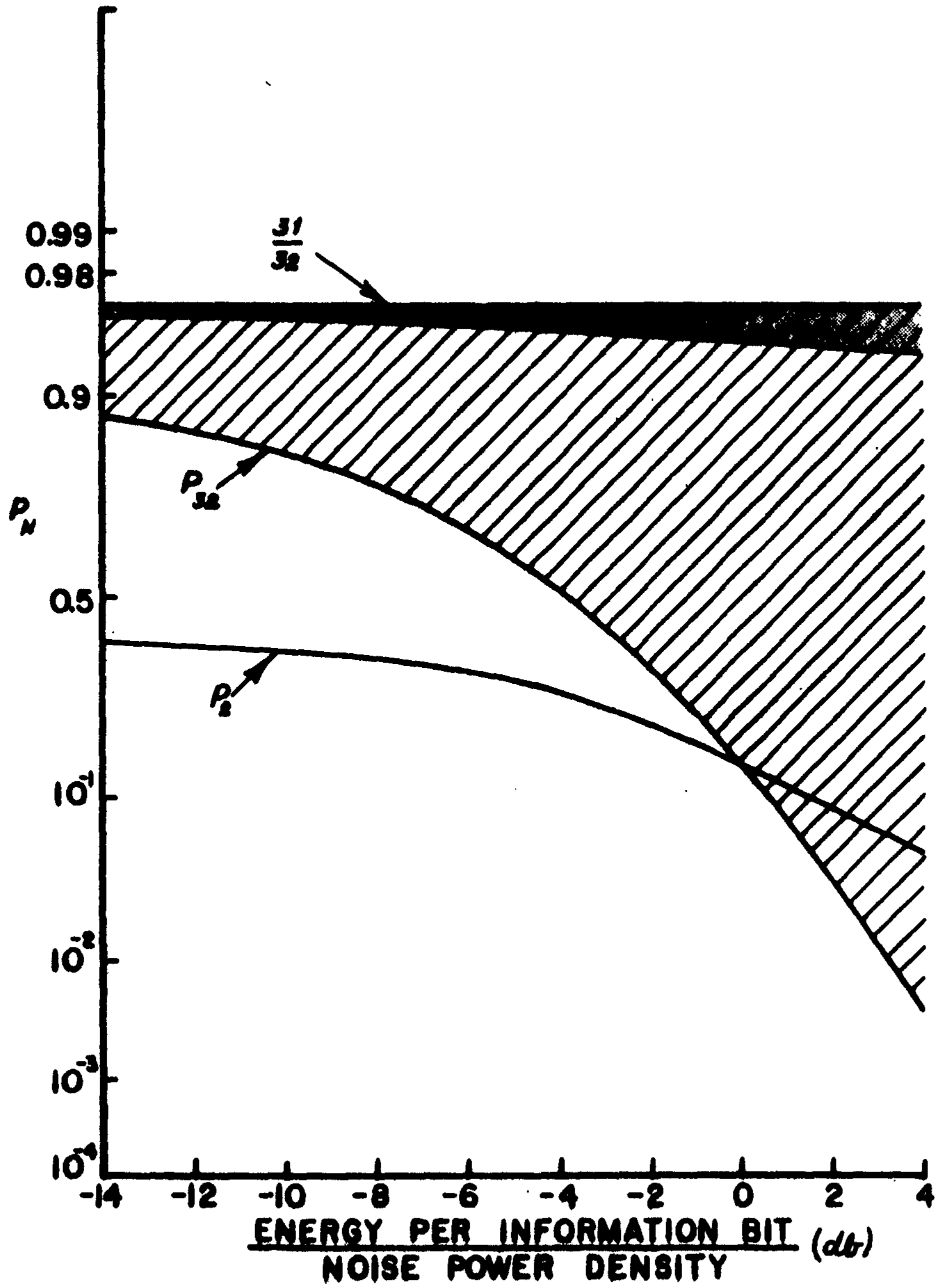

Figure 3 INCLUSION OF 32-ARY CHANNELS

Figure 4 depicts the situation for $N_1 = 32$ and $N_2 = 2$ and $K = 2$. The binary channels included by the given binary channel have error probabilities lying in the shaded region, whereas the binary channels included by the given 32-ary channel have error probabilities lying in the dashed region. The change in relative performance at an abscissa value of about -5 db is evident.

For the case of $N_1 < K < N_2$, it is found that C_{N_1, p_1} is more reliable than C_{N_2, p_2} if

$$\frac{K - N_1}{K - 1} + \frac{N_1}{K - 1} P_{N_1} \leq \frac{N_2}{N_2 - 1} P_{N_2} \tag{13}$$

which indicates a dependence upon K. This case has not been investigated in any further detail.

In summary, a method for comparing digital communication systems is presented which (1) encompasses channels represented by arbitrary transition probability matrices between input and output symbols, (2) relates directly to Shannon's partial ordering of channels in the sense that, if one channel includes a second, it is at least as reliable as the second independently of the size of the comparison alphabet, (3) duplicates Wolf's results for the special cases treated by the methods in his note.

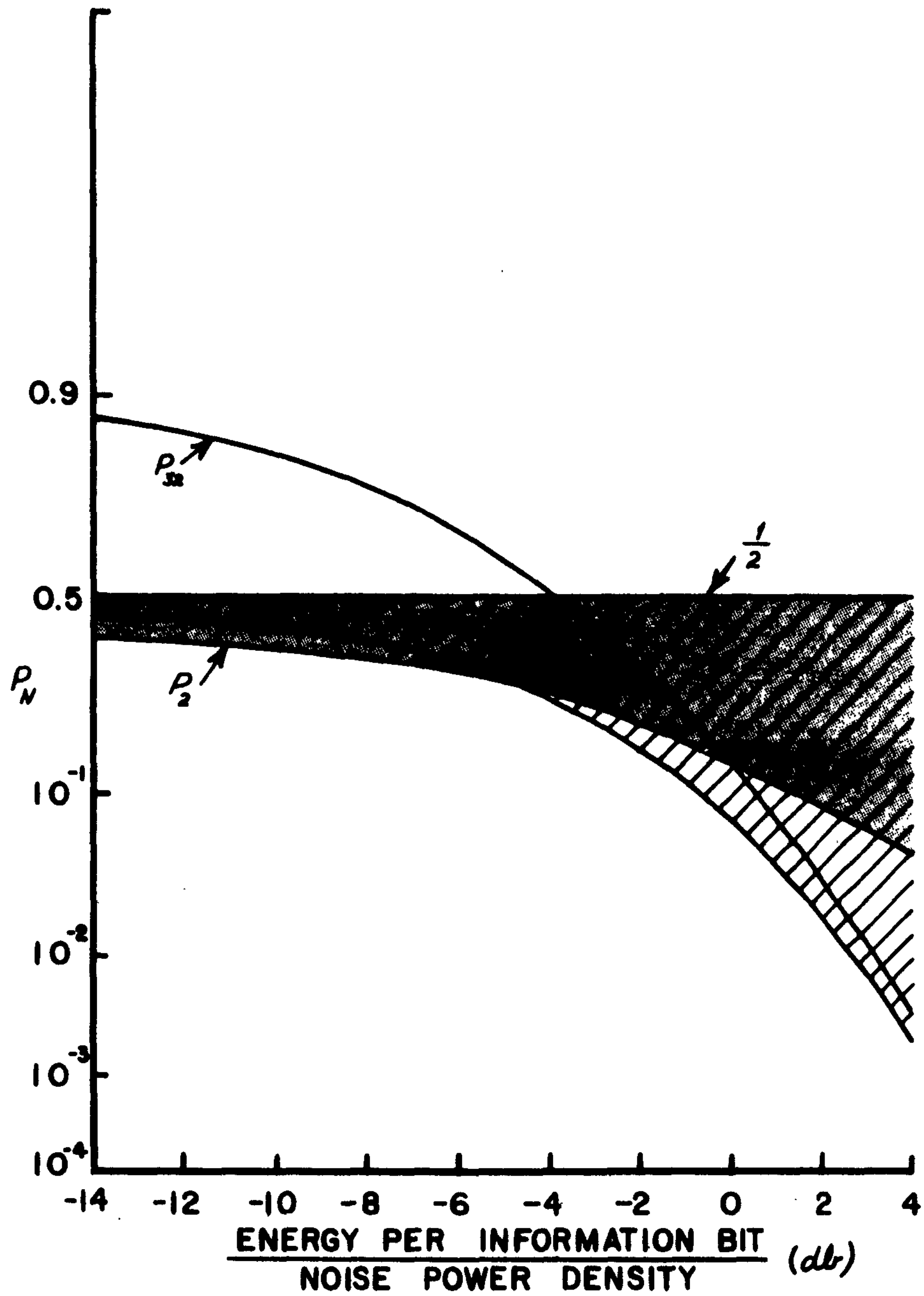

Figure 4 INCLUSION OF BINARY CHANNELS

APPENDIX: PROOF OF THEOREM

A proof of the result (7) is given here. Define a pure stochastic matrix as a stochastic matrix whose elements consist entirely of zeros and ones. Then, given any m by n stochastic matrix P, it is possible to express P as a convex linear combination of at most $m(n-1)+1$ pure stochastic matrices $P^{(i)}$, i.e.,

$$P = \sum_{i=1}^{\lambda} \omega_i P^{(i)}, \quad \omega_i > 0, \quad \sum_{i=1}^{\lambda} \omega_i = 1, \quad \lambda \leq m(n-1)+1 \tag{A-1}$$

To prove this, let $P = [p_{ij}]$ be any m by n stochastic matrix. Let j_i denote the column index of the minimum non-zero element in the i^{th} row; if this non-zero minimum occurs in more than one column, j_i may be selected as the index of any one such column. The non-zero row minimums are then the p_{ij_i}. Let

$$\omega_1 = \min_i \{p_{ij_i}\}$$

and let $P^{(1)}$ be the pure stochastic matrix defined by $p^{(1)}_{ij_i} = 1$. Consider the matrix, $P - \omega_1 P^{(1)}$ which differs from P only in the (i, j_i) elements, in which case the elements are

$$p_{ij_i} - \omega_1 \geq 0$$

Furthermore, at least one of the $p_{ij_i} - \omega_1$ is equal to zero. Thus, the matrix, $P - \omega_1 P^{(1)}$, has at least one more zero element than P. In addition, the row sums of $P - \omega_1 P^{(1)}$ all equal $1 - \omega_1$. Repeat the above procedure on the matrix, $P - \omega_1 P^{(1)}$, to obtain a second pure stochastic matrix, $P^{(2)}$, and consider the matrix, $P - \omega_1 P^{(1)} - \omega_2 P^{(2)}$. This matrix contains non-negative elements, has at least one more zero element than $P - \omega_1 P^{(1)}$, and has row sums all equal to $1 - \omega_1 - \omega_2$. This procedure is repeated as long as the resulting matrix contains a row with more than one non-zero element. Since there are only finitely many elements in the original matrix, P, and since each repetition of the above procedure produces a matrix with at least one more zero element than the preceding one, the process must terminate. Assume this occurs after r-1 repetitions. Then,

$$P - \omega_1 P^{(1)} - \omega_2 P^{(2)} - \cdots - \omega_{\lambda-1} P^{(\lambda-1)} = Q,$$

where each row of the matrix, Q, contains only one non-zero element equal to $1-\omega_1-\omega_2-\cdots-\omega_{k-1}$. Thus, Q has the form

$$Q=(1-\omega_1-\omega_2-\cdots-\omega_{k-1})P^{(k)}$$

where $P^{(k)}$ is a pure stochastic matrix. Setting $1-\omega_1-\omega_2-\cdots-\omega_{k-1}=\omega_k$,

$$P=\omega_1 P^{(1)}+\omega_2 P^{(2)}+\cdots \omega_k P^{(k)}$$

where the $P^{(i)}$ are pure stochastic matrices and $\omega_i>0, \sum_{i=1}^{k}\omega_i=1$ Thus, P is expressed as a convex linear combination of the above pure stochastic matrices. The maximum possible value of k is attained if each repetition in the above process produces a matrix with only one additional zero element and if there are initially no zero element in P. Therefore, $k\le m(n-1)+1$

Consider now any transformation of the type given by (5).

$$\sum_{\alpha} g_\alpha R_\alpha P T_\alpha = Q$$

If each R_α and T_α is replaced by its representation as a convex linear combination of pure stochastic matrices, the transformation is expressed in a form involving pre- and post-multiplication of P by pure stochastic matrices only. Thus, one need only consider transformations with the R_α and T_α pure stochastic matrices.

The stochastic matrices, $P=[p_{ij}]$ and $Q=[q_{ij}]$, defined by

$$p_{ij}=p, \quad i=j \qquad i,j=1,2,\cdots,N$$
$$=\frac{1-p}{N-1} \quad i\neq j$$

$$q_{ij}=t, \quad i=j \qquad i,j=1,2,\cdots,R$$
$$=\frac{1-t}{R-1}, \quad i\neq j$$

are contained in the symmetric channels, $C_{N,p}$ and $C_{R,t}$, respectively. We are interested in conditions under which $C_{N,p}\supseteq C_{R,t}$ or, equivalently, $P\supseteq Q$.

To determine necessary conditions, assume that $P \supseteq Q$. Then there exists a transformation such that

$$\sum_{\alpha} g_{\alpha} R_{\alpha} P T_{\alpha} = Q$$

The traces of the matrices are related as follows,

$$\sum_{\alpha} g_{\alpha} \, tr (R_{\alpha} P T_{\alpha}) = tr \, Q = R t \tag{A-2}$$

and, thus

$$\frac{1}{R} \min_{R_{\alpha}, T_{\alpha}} \left\{ tr \, (R_{\alpha} P T_{\alpha}) \right\} \leq t \leq \frac{1}{R} \max_{R_{\alpha}, T_{\alpha}} \left\{ tr \, (R_{\alpha} P T_{\alpha}) \right\} \tag{A-3}$$

The use of min and max is justified by the fact that only pure stochastic matrices, R_{α} and T_{α} , need be considered and, since these are finite in number, the minimum and maximum must occur for some specific R_{α} , T_{α} pairs in the set.

Suppose these pairs are R^- , T^- and R^+ , T^+ , i. e.,

$$\min_{R_{\alpha} T_{\alpha}} \left\{ tr (R_{\alpha} P T_{\alpha}) \right\} = tr (R^- P T^-) = tr \, Q^- \tag{A-4}$$

$$\max_{R_{\alpha}, T_{\alpha}} \left\{ tr \, (R_{\alpha} P T_{\alpha}) \right\} = tr (R^+ P T^+) = tr \, Q^+ \tag{A-5}$$

where

$$Q^- = R^- P T^- , \quad Q^+ = R^+ P T^+ \tag{A-4}$$

Equation (A-3) is a necessary condition that $C_{N,p} \supseteq C_{R,t}$. To show that it is also sufficient, assume that the condition is satisfied, i. e.,

$$\frac{1}{R} \, tr \, Q^- \leq t \leq \frac{1}{R} \, tr \, Q^+ \tag{A-7}$$

Let K_{Q^+} represent the channel containing Q^+ , and consider the transformation,

$$\sum_{\alpha=1}^{R!} \frac{1}{R!} R_{\alpha} Q^+ R_{\alpha} = U^+, \tag{A-8}$$

where the R_α range over all permutation matrices of order R. The elements of U^+ are

$$\mu_{ij}^+ = \frac{1}{R} \operatorname{tr} Q^+ \quad , \; i = j$$
$$= \frac{1 - 1/R \operatorname{tr} Q^+}{R-1} \quad , \; i \neq j \qquad i, j = 1, 2, \cdots R \tag{A-9}$$

Now $P \geqq Q^+$ by (A-6) and $Q^+ \geqq U^+$ by (A-8). Then $P \geqq U^+$ by property (iii) of (4).

In an entirely similar manner, one finds that $P \geqq U^-$ where

$$\mu_{ij}^- = \frac{1}{R} \operatorname{tr} Q^- \qquad i = j$$
$$= \frac{1 - 1/R \operatorname{tr} Q^-}{R-1} \qquad i \neq j \tag{A-10}$$
$$i, j = 1, 2, \cdots, R$$

From (A-7),

$$Q = \omega Q^+ + (1-\omega) Q^- \quad , \quad 0 \leq \omega \leq 1 \quad , \tag{A-11}$$

and, by a result of Shannon[4], $P \geqq Q$, which demonstrates that Condition (A-3) is sufficient for $C_{N,p} \geqq C_{R,t}$

It remains to explicitly evaluate the minimum and maximum contained in (A-3). Let $Q^{(\alpha)} = R_\alpha P T_\alpha$ and consider a typical element on the main diagonal of $Q^{(\alpha)}$,

$$q_{kk}^{(\alpha)} = \sum_{\nu=1}^{N} \sum_{\mu=1}^{N} r_{k\nu}^{(\alpha)} p_{\nu\mu} t_{\mu k}^{(\alpha)} = \sum_{\mu=1}^{N} p_{\nu_k^\alpha \mu} t_{\mu k}^{(\alpha)} \; ,$$

where ν_k^α is the index of the column of R_α in which the "k"th row unity appears. The effect of R_α in determining $q_{kk}^{(\alpha)}$ is to select out the ν_k^α row of P. Now, let J_k^α be the set of row indices of T_α corresponding to the rows in which the "K"th column unities of T_α appear. Then $J_1^\alpha, J_2^\alpha, \cdots, J_R^\alpha$ is a partitioning of the row indices, 1, 2, ..., N, into R mutually disjoint sets, some of which may be empty. Then

$$q_{kk}^{(\alpha)} = \sum_{\mu \in J_k^\alpha} p_{\nu_k^\alpha \mu}$$

and

$$\operatorname{tr} Q^\alpha = \sum_{k=1}^{R} q_{kk}^{(\alpha)} = \sum_{k=1}^{R} \sum_{\mu \in J_k^\alpha} p_{\nu_k^\alpha \mu}$$

In forming $\mathrm{tr}\, Q^{(\alpha)}$, one is free to select a single element from each column of P, the sum of these elements being $\mathrm{tr}\, Q^{(\alpha)}$. In addition, these elements can be selected from not more than R rows since the range of the dummy index, k, is from 1 to R.

For $R \geq N$, each row of P can be utilized and $\mathrm{tr}\, Q^{(\alpha)}$ is maximized by selecting the maximal element from each column. If $p > 1/N$ then $p > (1-p)/(N-1)$, and the maximal column elements lie on the main diagonal of P. On the other hand, if $p < 1/N$, then $p < (1-p)/(N-1)$ and each column of P contains N-1 maximal elements, each equal to $(1-p)/(N-1)$. Thus, for $R \geq N$

$$
\begin{aligned}
\max(\mathrm{tr}\, Q^{(\alpha)}) &= Np \quad \text{IF} \quad p > 1/N \\
&= N\,\frac{1-p}{N-1} \quad \text{IF} \quad p < 1/N
\end{aligned}
$$

If $p = \frac{1}{N} = \frac{1-p}{N-1}$, then all elements of P are equal and

$$
\max(\mathrm{tr}\, Q^{(\alpha)}) = Np = N\,\frac{1-p}{N-1} = 1
$$

To find min$(\mathrm{tr}\, Q^{(\alpha)}$ the argument proceeds exactly as above except that, in this case, minimal column elements of P are considered. One obtains

$$
\begin{aligned}
\min\left\{\mathrm{tr}\, Q^{(\alpha)}\right\} &= N\,\frac{1-p}{N-1} \qquad \text{IF} \quad p \geq \frac{1}{N} \\
&= Np \qquad \text{IF} \quad p \leq \frac{1}{N}
\end{aligned}
$$

Thus, for $R \geq N$, Equation (A-3) can be written

$$
\frac{1-p}{N-1}\,\frac{N}{R} \leq t \leq p\,\frac{N}{R}, \qquad \text{IF} \quad p \geq \frac{1}{N},
$$

$$
p\,\frac{N}{R} \leq t \leq \frac{1-p}{N-1}\,\frac{N}{R}, \qquad \text{IF} \quad p \leq \frac{1}{N}
$$

which is in agreement with the first part of (7).

Consider now the case of $R \leq N$. The above argument must be modified in that only R rows of P may be utilized in the maximization and minimization of $\mathrm{tr}\, Q^{(\alpha)}$. Thus, one must select a representative from each column of P with the restriction that at most R rows may be utilized. In maximizing $\mathrm{tr}\, Q^{(\alpha)}$ if $p > \frac{1}{N}$, each of the R rows can contribute one "p" but the elements taken from the remaining $N-R$ columns must then equal $\frac{1-p}{N-1}$. On the other hand, if $p < \frac{1}{N}$, maximization is accomplished by selecting a "$\frac{1-p}{N-1}$" from each column, which is possible since $R > 1$. Thus,

$$\max \left\{ tr\, Q^{(\alpha)} \right\} = Rp + (N-R)\frac{1-p}{N-1}$$

$$= \frac{R-1}{N-1} Np + \frac{N-R}{N-1} \qquad \text{IF } p > \frac{1}{N}$$

and

$$\max \left\{ tr\, Q^{(\alpha)} \right\} = N \frac{1-p}{N-1} \qquad \text{IF } p < \frac{1}{N}$$

For $p = \frac{1}{N}$, these maxima are identical. Similar considerations apply in finding $\min(tr\, Q^{(\alpha)})$. If $p > \frac{1}{N}$, the $\frac{1-p}{N-1}$ elements are minimal column elements and N of these can be selected. On the other hand, if $p < \frac{1}{N}$, the "p"s are minimal column elements and only R of these can be selected, the remaining $N-R$ column representatives necessarily equaling $(1-p)/(N-1)$. Thus,

$$\min \left\{ tr\, Q^{(\alpha)} \right\} = N \frac{1-p}{N-1}, \qquad \text{IF } p \geq \frac{1}{N}$$

$$= \frac{R-1}{N-1} Np + \frac{N-R}{N-1}, \quad \text{IF } p \leq \frac{1}{N}.$$

Thus, for $R \leq N$, Equation (A-3) can be written

$$\frac{1-p}{N-1} \frac{N}{R} \leq t \leq \frac{R-1}{N-1} \frac{N}{R} p + \frac{N-R}{R(N-1)} \qquad \text{IF } p \geq \frac{1}{N}$$

$$\frac{R-1}{N-1} \frac{N}{R} p + \frac{N-R}{R(N-1)} \leq t \leq \frac{1-p}{N-1} \frac{N}{R} \qquad \text{IF } p \leq \frac{1}{N}$$

which is in agreement with the second part of (7).

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VII

OPTIMIZATION OF DIGITAL COMMUNICATION SYSTEMS OPERATING OVER A DISPERSIVE CHANNEL

OPTIMIZATION IN THE FREQUENCY DOMAIN

SUMMARY

This analysis is concerned with optimization in the sense of minimizing probability of error of a digital communication system, where we have control over both the transmitter waveforms and the receiving system but not over the channel transfer function or the noise properties. The transmitted signals are assumed to occur independently and with equal probabilities. The energy and duration of the transmitted signals are specified. The noises added at the input and output of the dispersive channel are assumed gaussian, but not necessarily white; hence, a linear receiver is used. Matrix Equations (10) and (11) give the relationships which must exist in an optimum system among the signal, receiver, channel and noise functions. These equations can be readily solved for the optimum receiver given the transmitted waveforms and vice versa. The main problem is, however, to optimize both the waveform and the receiver simultaneously. For a particular situation, i.e., specified channel transfer function and noise autocorrelation function, the form of the solution is obtained. That is, series expressions for the optimum transmitted waveforms and the impulse responses of the receiving filters are developed. The coefficients of the series have, however, been specified only for the binary case. Interesting orthogonality properties which the component functions possess are developed.

The chapter concludes with an alternate representation of the probability of error based on geometric concepts.

INTRODUCTION

Figure 1 illustrates the system to be analyzed.*

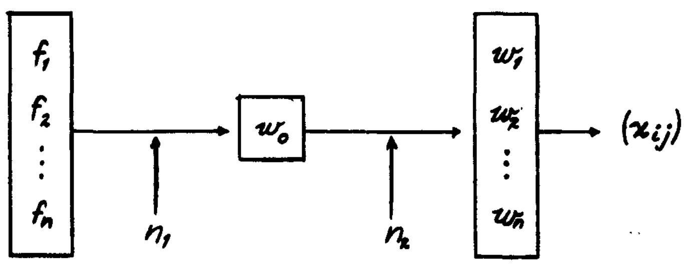

FIGURE 1

* This system was previously considered in Reference 1 where a time domain analysis was employed instead of the frequency domain analysis which is used here.

The input signals are time limited, i.e., the f_i vanish outside an interval $[0,T]$ and it is assumed as well that the filter functions w_i are zero for $t > T$. The noises n_1, n_2 are additive, gaussian with zero mean.

The filters are sampled at $t - T$ and the decision made that f_i was sent if the output of w_i exceeds the output of each of the other filters. In Reference 1, an expression was obtained for the probability of correct decision P_c as a functional of the f_i, w_i. The problem was posed of determining the set of functions, signals and filters which maximize P_c when w_0 and the noise correlations are given and the energies of the signals limited. A set of necessary conditions on the f_i, w_i was obtained by means of the variational calculus. These conditions had the superficial appearance of a system of integral equations but the kernels were, themselves, functionals of the unknown. An explicit solution was then obtained, including the calculation of P_c, for a particular w_0 and noise correlation in the case $N = 2$. It is found, however, that the techniques which succeeded for $N = 2$ were intractable for larger N.

The present investigation deals with the same set of necessary conditions on the time functions w_i, f_i. By replacing these conditions by equivalent ones on the Fourier transforms and operating in the transform domain, we have succeeded in finding the form of the w_i, f_i for general N (where the same w_0 and correlation as previously used have been retained). This is the main result and is given in Sections 2 and 3. The f_i, w_i are found to be linear combinations of functions of the same class that solved the case $N = 2$. But, as yet, we have not been able to determine the coefficients which complete the solution for $N > 2$.

In Section 4, some orthogonality relations are given which the functions arising in Section 3 satisfy, along with some invariance properties of the basic equations in the time domain. These results are important for the construction of explicit solutions from the general form.

An alternate representation is obtained in Section 5 of the conditional probabilities of correct decision which, in some respects, is more convenient than that given in Reference 1.

TRANSFORM DOMAIN, PRELIMINARY THEORY

We consider functions of a complex variable of the form

$$G(\omega) = C \quad \frac{e^{-iT(\omega+\beta)} - 1}{\omega+\beta} \tag{1}$$

where T is real and C , β are real or complex.

Expanding the exponential in (1), we observe that the singularity at $\omega = -\beta$ is removable. Defining $G(\omega)$ by continuity at this point, we have that $G(\omega)$ is entire. It is shown, as follows, that the IFT (inverse Fourier transform) of $G(\omega)$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left[\frac{e^{-iT(\omega+\beta)} - 1}{\omega+\beta} \right] d\omega \tag{2}$$

vanishes outside the interval $[0, T]$, where real t is understood and when T is negative $[0, T]$ is understood to mean $[T, 0]$. The assertion is evident if $T=0$. For $T > 0$, if $t < 0$ or $t > T$, we can write

$$g(t) = \frac{1}{2\pi} \int_C e^{i\omega t} \left[\frac{e^{-iT(\omega+\beta)} - 1}{\omega+\beta} \right] d\omega \tag{3}$$

where C is the contour consisting of the real line completed by a large semicircle in the LHP (lower half plane) in the first case, or by a large semicircle in the UHP (upper half plane) in the second. Hence, $g(t)$ vanishes in either case, and similarly if $T < 0$.

For any function of the form

$$u(\omega) = \frac{e^{-iT\omega}}{\omega+\beta} \quad , \quad T \text{ real} \tag{4}$$

we define

$$[u(\omega)]_R = \frac{e^{iT\beta}}{\omega+\beta} \quad , \tag{5}$$

i.e., the ω in the exponent is replaced by the zero of the denominator. And for any linear combination of functions, $u_i(\omega)$ of the form (4), we define

$$\left[\sum_i C_i u_i(\omega)\right]_R = \sum_i C_i \left[u_i(\omega)\right]_R \tag{6}$$

We observe that the function

$$\frac{e^{-iT\omega}}{\omega+\beta} - \left[\frac{e^{-iT\omega}}{\omega+\beta}\right]_R \tag{7}$$

has the form (1) and that any linear combination of such functions, T fixed, has an IFT which vanishes outside $[0,T]$.

If $R(\omega) = N(\omega)/D(\omega)$ is a rational function for which degree of $N <$ degree D and D has only simple zeros, we see from the partial fraction representation that

$$e^{-i\omega T} R(\omega) - \left[e^{-i\omega T} R(\omega)\right]_R \tag{8}$$

has an IFT which vanishes outside $[0,T]$.

In evaluating $[e^{-i\omega T} R(\omega)]_R$, it is not necessary to operate in each case with the explicit partial fraction representation, for we have that if β_i , i = 1, 2, ...M are the zeros of D then

$$\left[e^{-i\omega T} R(\omega)\right]_R = \sum_{i=1}^{n} \frac{e^{-iT\beta_i} N(\beta_i)}{(\omega-\beta_i) D_i(\beta_i)} \tag{9}$$

where $$D_i(\omega) = \frac{D(\omega)}{\omega-\beta_i}$$

GENERAL SOLUTION OF MATRIC EQUATIONS IN TRANSFORM DOMAIN

Equations (30) and (31) of Reference 1 gave the necessary conditions that the probability of correct decision, P_c , be a maximum. These may be written in matric form.

$$\nu \underset{\sim}{h}(T-\tau) - \lambda \underset{\sim}{f}(\tau) = \underset{\sim}{0} \tag{10}$$

$$\tau \in [0,T]$$

$$\nu \underset{\sim}{g}(T-\tau) - \eta \underset{\sim}{r}(\tau) = \underset{\sim}{0} \tag{11}$$

where ν , η , λ are constant square matrices and $\underset{\sim}{h}$, $\underset{\sim}{f}$, $\underset{\sim}{g}$, $\underset{\sim}{r}$ are N dimensional vector valued functions of time (column matrices) which are distinguished by the tilde written beneath. The transpose of ν is indicated by $\tilde{\nu}$. The interval on which (10) and (11) are required to hold follows from the requirement (which we impose as before) that the N signals f_i and the N weighting functions w_i vanish outside the interval $[0,T]$. The vectors $\underset{\sim}{h}$, $\underset{\sim}{g}$, $\underset{\sim}{r}$ are defined by

$$\underset{\sim}{g} = w_0 * \underset{\sim}{f} = \begin{bmatrix} w_0 * f_1 \\ w_0 * f_2 \\ \vdots \\ w_0 * f_N \end{bmatrix} \tag{12}$$

$$\underset{\sim}{h}_1 = w_0 * \underset{\sim}{w} \tag{13}$$

$$\underset{\sim}{r} = \rho * \underset{\sim}{w} \tag{14}$$

where w_0 and ρ are the channel weighting function and noise correlation function, as in Reference 1. The square matrices have the properties

$$\lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{bmatrix} \tag{15}$$

where the λ_l are Lagrangian multipliers,

$$\sum_l \nu_{kl} = 0 \,, \quad k = 1, 2, \ldots N \tag{16}$$

$$\sum_l \eta_{kl} = 0 \,, \quad k = 1, 2, \ldots N \tag{17}$$

$$\eta = \tilde{\eta} \,. \tag{18}$$

Since (16) and (17) imply that ν , η are singular, it is convenient to make the following transformations. Define η_0 , ν_0 by

$$\begin{aligned} \eta &= \tilde{J}_0 \, \eta_0 \, J_0 \\ \nu &= \nu_0 \, J_0 \end{aligned} \,. \tag{19}$$

(These transformations will permit us to modify a certain matrix product which will occur subsequently so that it will not be singular.)

where

$$J_0 = \begin{bmatrix} 1, 0, 0, \cdots\cdots -1 \\ 0, 1, 0, \cdots\cdots -1 \\ 0, 0, 1, 0, \cdots -1 \\ \vdots \\ 0, 0, 0 \cdots\cdots 1, -1 \\ 0, 0, 0 \cdots\cdots\cdots 1 \end{bmatrix}, \quad J_0^{-1} = \begin{bmatrix} 1, 0, 0, & & 1 \\ 0, 1, 0, 0 & & 1 \\ 0, 0, 1, 0 & & 1 \\ & \ddots & \vdots \\ 0, 0, & & 1 \end{bmatrix}. \tag{20}$$

Then η_0 , ν_0 have the form

$$\eta_0 = \begin{bmatrix} \eta_{11}, \eta_{12}, \cdots \eta_{1,N-1}, 0 \\ \eta_{21}, \eta_{22}, \cdots \eta_{2,N-1}, 0 \\ \vdots \\ \eta_{N-1,1} \cdots \eta_{N-1,N-1}, 0 \\ 0, 0, \cdots\cdots 0 \end{bmatrix}, \qquad \nu_0 = \begin{bmatrix} \nu_{11}, \nu_{12}, \cdots \nu_{1,N-1}, 0 \\ \nu_{21}, \nu_{22}, \cdots \nu_{2,N-1}, 0 \\ \vdots \qquad \vdots \\ \nu_{N1}, \nu_{N2}, \cdots \nu_{N,N-1}, 0 \end{bmatrix}. \tag{21}$$

Upon substituting for ν , η in (10) and (11) and multiplying the resulting (11) from left by $\tilde{J}_0^{-1}$, we get

$$\nu_0 \underset{\sim}{h}'(T-\tau) - \lambda \underset{\sim}{f}(\tau) = \underset{\sim}{0} \tag{22}$$

$$\tilde{\nu}_0 \underset{\sim}{g}(T-\tau) - \eta_0 \underset{\sim}{r}'(\tau) = \underset{\sim}{0} \tag{23}$$

where

$$\begin{aligned} \underset{\sim}{h}' &= J_0 h = J_0 w_0 * \underset{\sim}{w} = w_0 * J_0 \underset{\sim}{w} = w_0 * \underset{\sim}{w}' \\ \underset{\sim}{r}' &= J_0 \underset{\sim}{r} = J_0 \rho * \underset{\sim}{w} = \rho * J_0 \underset{\sim}{w} = \rho * \underset{\sim}{w}', \\ \underset{\sim}{w}' &= J_0 \underset{\sim}{w} \end{aligned} \tag{24}$$

e.g., $N = 4$

$$\underset{\sim}{w}' = \begin{bmatrix} w_1 - w_4 \\ w_2 - w_4 \\ w_3 - w_4 \\ w_4 \end{bmatrix}$$

Writing explicitly in terms of the unknowns, we have

$$\nu_0 (w_0 * \underset{\sim}{w}')_{T-\tau} - \lambda \underset{\sim}{f}\, \tau = \underset{\sim}{0} \tag{25}$$

$$\tilde{\nu}_0 (w_0 * \underset{\sim}{f})_{T-\tau} - \eta_0 (\rho * \underset{\sim}{w}')\, \tau = \underset{\sim}{0} \tag{26}$$

(We shall at times for convenience indicate arguments of certain functions by subscripts.)

Our problem is to find real functions $\underset{\sim}{w}'$, $\underset{\sim}{f}$, when w_0, ρ are given, which satisfy (25), (26) in the interval $[0, T]$ and which vanish outside this interval. We define the FT (Fourier transform) of a real function $u(t)$ by

$$U(w) = \int_{-\infty}^{\infty} e^{-i\omega t}\, u(t)\, dt$$

and note that if $w(t) = u(T-t)$, then $W(\omega) = e^{-i\omega T}\, \bar{U}(\omega)$

Let the left sides of (25), (26) be denoted by

$$\underset{\sim}{u}(\tau) \equiv \nu_0 \,(w_0 * \underset{\sim}{w}')_{T-\tau} - \lambda \underset{\sim}{f}\, \tau \tag{27}$$

$$\underset{\sim}{v}(\tau) \equiv \tilde{\nu}_0 \,(w_0 * \underset{\sim}{f})_{T-\tau} - \eta_0 \,(\rho * \underset{\sim}{w}')\, \tau \tag{28}$$

Using the convolution theorem together with the property just noted gives

$$\underset{\sim}{U}(\omega) = \nu_0 \,(\overline{W_0 \underset{\sim}{W}'})\, e^{-i\omega T} - \lambda \underset{\sim}{F} \tag{29}$$

$$\underset{\sim}{V}(\omega) = \tilde{\nu}_0 \,(\overline{W_0 \underset{\sim}{F}})\, e^{-i\omega T} - \eta_0\, \Phi \underset{\sim}{W}' \tag{30}$$

where the bar indicates complex conjugate and W_0, $\underset{\sim}{W}'$, $\underset{\sim}{F}$, Φ are the FT's of w_0, $\underset{\sim}{w}'$, $\underset{\sim}{f}$, ρ. We may now rephrase our problem as that of finding $\underset{\sim}{F}$, $\underset{\sim}{W}'$ so that the IFT's, $\underset{\sim}{f}$, $\underset{\sim}{w}'$ are real and vanish outside $[0, T]$ while the IFT's $\underset{\sim}{u}$, $\underset{\sim}{v}$ of (27), (28) vanish inside $[0, T]$.

Let us take

$$\underset{\sim}{F} = \lambda^{-1} \nu_0\, e^{-i\omega T}\, \bar{\underset{\sim}{H}}' - \left[\lambda^{-1} \nu_0\, e^{-i\omega T}\, \bar{\underset{\sim}{H}}'\right]_R \tag{31}$$

where $\underset{\sim}{H}' = W_0 \underset{\sim}{W}'$ and the operator $[\cdot]_R$ applied to the column matrix means that the operator is to be applied to each element. It is understood that we shall subsequently take $\underset{\sim}{H}'$ in a form consistent with Section 2. Then $\underset{\sim}{F}$ represents (if the FT of) a function f which vanishes outside $[0, T]$. This does not yet make f real, but we note that since w_0 is real, f will be real if $\underset{\sim}{w}'$ is real. We use here the criterion that the function represented is real if $\bar{W}(\omega) = W(-\omega)$. Applied to (31) together with the fact that $[\cdot]_R$ commutes with conjugation and, also, with sign change of ω, we have

$$\bar{\underset{\sim}{F}} = \lambda^{-1} \nu_0\, e^{i\omega T}\, \underset{\sim}{H}' - \lambda^{-1} \nu_0 \left[e^{i\omega T} \underset{\sim}{H}'\right]_R . \tag{32}$$

Then

$$\tilde{\nu}_0 e^{-i\omega T} \overline{W}_0 \underset{\sim}{F} = \tilde{\nu}_0 \lambda^{-1} \nu_0 \overline{W}_0 \left(\underset{\sim}{H}' - e^{-i\omega T}\left[e^{i\omega T}\underset{\sim}{H}'\right]_R\right). \tag{33}$$

and so (30) can be written

$$\underset{\sim}{V}(\omega) = (\tilde{\nu}_0 \lambda^{-1} \nu_0 W_0 \overline{W}_0 - \eta_0 \phi)\underset{\sim}{W}' - \tilde{\nu}_0 \lambda^{-1} \nu_0 e^{-i\omega T} \overline{W}_0 \left[e^{i\omega T} W_0 \underset{\sim}{W}'\right]_R. \tag{34}$$

Now the elements of the last row and column of the matrix

$$\Psi = \tilde{\nu}_0 \lambda^{-1} \nu_0 \tag{35}$$

are all zeros. We define Ψ_0 as the $(N-1)$ by $(N-1)$ matrix obtained from Ψ by deleting the last row and column, and we denote by η_{00} the $(N-1)$ by $(N-1)$ matrix obtained from η_0 in the same way. Putting

$$\gamma = \Psi_0^{-1} \eta_{00} \tag{36}$$

(the existence of the inverse may be assumed), we have

$$\underset{\sim}{V} = \Psi_0 \left[(W_0 \overline{W}_0 I - \gamma \phi)\underset{\sim}{W}' - e^{-i\omega T} \overline{W}_0 (e^{i\omega T} W_0 \underset{\sim}{W}')_R\right]. \tag{37}$$

We now take $\underset{\sim}{W}'$ in the form

$$\underset{\sim}{W}' = \underset{\sim}{W}_a - e^{-i\omega T}\left[e^{i\omega T}\underset{\sim}{W}_a\right]_R, \quad T > 0 \tag{38}$$

where $\underset{\sim}{W}_a$ will subsequently be taken in the appropriate rational function form. Writing (38) as

$$\underset{\sim}{W}' = \left[\underset{\sim}{W}_a e^{i\omega T} - (e^{i\omega T}\underset{\sim}{W}_a)_R\right] e^{-i\omega T}$$

we see by Section 2 that the square bracket represents a function which vanishes outside $[-T, 0]$ and so $\underset{\sim}{W}'$ itself represents a function which vanishes outside $[0, T]$. Using (38), $\underset{\sim}{V}$ becomes

$$\underset{\sim}{V} = \Psi_0 \Big\{(W_0 \overline{W}_0 I - \gamma \phi)\underset{\sim}{W}_a - (W_0 \overline{W}_0 I - \gamma \phi) e^{-i\omega T}(e^{i\omega T}\underset{\sim}{W}_a)_R \tag{39}$$
$$- e^{-i\omega T} \overline{W}_0 \left[e^{i\omega T} W_0 \underset{\sim}{W}_a - W_0 (e^{i\omega T}\underset{\sim}{W}_a)_R\right]_R\Big\}.$$

We have to find $\underset{\sim}{W}_a$ so that $\underset{\sim}{W}'$ represents a real function and $\underset{\sim}{U}$, $\underset{\sim}{V}$ represent functions which vanish in $[0, T]$. To illustrate the technique that may be applied to this end, we consider the case

$$W_0 = \frac{a}{a + i\omega} \tag{40}$$

$$\Phi(\omega) = \eta_2 + \frac{\eta_1 a^2}{a^2 + \omega^2} \qquad \text{(white noise at input \& output of } \omega_0 \text{)} \tag{41}$$

The first term of (39) is (except for the matrix factor Ψ_0 which will be seen to be inessential to the following arguments concerning the vanishing of u, v in $[0, T]$)

$$\left[I - \frac{r\Phi}{W_0 \overline{W}_0}\right] W_0 \overline{W}_0 \underset{\sim}{W}_a$$

where

$$\frac{\Phi}{W_0 \overline{W}_0} = n_2 \frac{\omega^2}{a^2} + n_1 + n_2 \equiv \xi \tag{42}$$

Let $\underset{\sim}{W}_a$ be such that

$$\left[I - \frac{\gamma\Phi}{W_0 \overline{W}_0}\right] W_0 \overline{W}_0 \underset{\sim}{W}_a = \underset{\sim}{b}\, \overline{W}_0 \tag{43}$$

where $\underset{\sim}{b}$ is a constant vector.

Then

$$\underset{\sim}{W}_a = \frac{1}{W_0}\left[I - \frac{\gamma\Phi}{W_0 \overline{W}_0}\right]^{-1} \underset{\sim}{b} = \frac{1}{W_0}(I - \gamma\xi)^{-1} \underset{\sim}{b} \,. \tag{44}$$

Since W_0 represents a real function which vanishes for negative t, $\overline{W}_0$ represents a real function which vanishes for positive t and therefore in particular vanishes in $[0, T]$. Because of (43), each component of the first term in the curley bracket of (39) vanishes in $[0, T]$. To complete the requirement that v vanish in $[0, T]$, we shall choose the second and third terms of (39) so that their sum has no pole in the LHP. For the example, there is a pole in the LHP at $\omega = -ia$ and, in a moment, we shall impose the condition that the residues of these terms cancel. At the same time, we shall satisfy the requirement that u vanishes in $[0, T]$.

Let ξ_i be the roots of the polynomial

$$|\gamma \xi_i - I| = 0 \tag{45}$$

i. e., the ξ_i are the reciprocals of the eigenvalues of γ. We assume that the ξ_i are distinct so that we may use the representation for a function of a matrix

$$f(\gamma) = \sum_{i=1}^{N-1} f\left(\frac{1}{\xi_i}\right) \epsilon_i \tag{46}$$

where the ϵ_i are the resolutions of the identity (projection operators) which have the properties

$$\epsilon_i \epsilon_j = 0, \quad i \neq j$$
$$\epsilon_i^2 = \epsilon_i \qquad i = 1, 2, \cdots N-1$$
$$\sum_1^{N-1} \epsilon_i = I .$$

(For an arbitrary vector X, the matrix operation $\epsilon_i X$ projects X onto the subspace spanned by the eigenvectors associated with the eigenvalue ξ_i^{-1}.)

Thus, by (46), we may write

$$(I - \gamma \xi)^{-1} = \sum_1^{N-1} \frac{1}{1 - \xi/\xi_i} \epsilon_i = \sum_1^{N-1} \frac{\xi_i}{\xi_i - \xi} \epsilon_i$$
$$I - \gamma \xi = \sum_1^{N-1} \left(1 - \frac{\xi}{\xi_i}\right) \epsilon_i .$$

Hence, from (44)

$$\underset{\sim}{W_a} = \frac{1}{W_0} \sum_1^{N-1} \frac{\xi_i}{\xi_i - \xi} \epsilon_i \underset{\sim}{b} . \tag{47}$$

From (42)

$$\xi - \xi_i = n_2 \frac{w^2}{a^2} + n_1 + n_2 - \xi_i$$

and since the ξ_i are constant, we may introduce new constants β_i by

$$\frac{n_1 + n_2 - \xi_i}{n_2} = \frac{-\beta_i^2}{a^2} . \tag{48}$$

Thus,

$$\underset{\sim}{W_a} = -a(a + i\omega) \sum_1^{N-1} \frac{\xi_i}{n_2 (w^2 - \beta_i^2)} \epsilon_i \underset{\sim}{b} . \tag{49}$$

With the use of Section 2 and the linearity of the operator $[\cdot]_R$, we have

$$\left[e^{i\omega T} W_0 \underset{\sim}{W_a}\right]_R = -\sum_1^{N-1} a^2 \frac{\xi_i}{n_2} \epsilon_i \underset{\sim}{b} \left[\frac{1}{\omega-\beta_i} \quad \frac{e^{i\beta_i T}}{2\beta_i} + \frac{1}{\omega+\beta_i} \quad \frac{e^{-i\beta_i T}}{-2\beta_i}\right] \tag{50}$$

$$\left[e^{i\omega T} \underset{\sim}{W_a}\right]_R = -\sum_1^{N-1} a \frac{\xi_i}{n_2} \epsilon_i \underset{\sim}{b} \left[\frac{a+i\beta_i}{\omega-\beta_i} \quad \frac{e^{i\beta_i T}}{2\beta_i} + \frac{a-i\beta_i}{\omega+\beta_i} \quad \frac{e^{-i\beta_i T}}{-2\beta_i}\right] \tag{51}$$

$$\left[W_0 \left[e^{i\omega T} \underset{\sim}{W_a}\right]_R\right]_R = -\sum_1^{N-1} a^2 \frac{\xi_i}{n_2} \epsilon_i \underset{\sim}{b} \left[\frac{1}{\omega-\beta_i} \quad \frac{e^{i\beta_i T}}{2\beta_i} + \frac{1}{\omega+\beta_i} \quad \frac{e^{-i\beta_i T}}{-2\beta_i}\right.$$

$$\left. + \frac{1}{a+i\omega}\left[\frac{a+i\beta_i}{a_i-\beta_i} \quad \frac{e^{i\beta_i T}}{2\beta_i} + \frac{a-i\beta_i}{a_i+\beta_i} \quad \frac{e^{-i\beta_i T}}{-2\beta_i}\right]\right] \tag{52}$$

Hence, the third term in the curley bracket of (39) is

$$-e^{-i\omega T} \overline{W}_0 \left[\frac{1}{a+i\omega} \sum_1^{N-1} a^2 \frac{\xi_j}{n_2} \epsilon_j \underset{\sim}{b} \left(\frac{1}{i}\right)\left(\frac{e^{i\beta_j T}}{2\beta_j} - \frac{e^{-i\beta_j T}}{2\beta_j}\right)\right]. \tag{53}$$

We observe that the term in the square bracket of this expression is just $\underset{\sim}{U}(\omega)$ from which it follows that $\underset{\sim}{u}$ vanishes for all positive t and, therefore, in particular for $t \in [0, T]$. For the second term of (39), we have

$$-e^{-i\omega T} W_0 \overline{W}_0 \sum_{k=1}^{N-1} \left(\frac{-\xi_k + \xi}{\xi_{0k}}\right) \epsilon_k \sum_{j=1}^{N-1} a \frac{\xi_j}{n_2} \epsilon_j \underset{\sim}{b} \left[\frac{u+i\beta_j}{\omega-\beta_j} \frac{e^{i\beta_j T}}{2\beta_j} + \frac{a-i\beta_j}{\omega+\beta_j} \frac{e^{-i\beta_j T}}{-2\beta_j}\right] \tag{54}$$

Since $\epsilon_j \epsilon_k = 0$, $j \neq k$, the last expression becomes

$$-e^{-i\omega T} W_0 \overline{W}_0 \sum_1^{N-1} \frac{(\omega^2-\beta_j^2)}{a} \epsilon_j \underset{\sim}{b} \left[\frac{a+i\beta_j}{\omega-\beta_j} \quad \frac{e^{i\beta_j T}}{2\beta_j} + \frac{u-i\beta_j}{\omega+\beta_j} \quad \frac{e^{-i\beta_j T}}{-2\beta_j}\right]. \tag{55}$$

For the condition on the residues, we get

$$\sum_{j=1}^{N-1} \frac{(-a^2-\beta_j^2)}{a} \epsilon_j \underset{\sim}{b} \left[\frac{a+i\beta_j}{-a-\beta_j} \frac{e^{i\beta_j T}}{2\beta_j} + \frac{a-i\beta_j}{-ai+\beta_j} \frac{e^{-i\beta_j T}}{-2\beta_j} \right] \tag{56}$$

$$+\sum_{1}^{N-1} a\frac{\xi_j}{n_2} \epsilon_j \underset{\sim}{b} \left(\frac{1}{i}\right) \left[\frac{e^{i\beta_j T}}{2\beta_j} - \frac{e^{-i\beta_j T}}{2\beta_j} \right] = \underset{\sim}{0} \,.$$

Since the ϵ_j are disjoint, this implies that the coefficient of each ϵ_j must be zero. Hence,

$$i(a^2+\beta_j^2) \left[\frac{a+i\beta_j}{a-i\beta_j} e^{i\beta_j T} - \frac{a-i\beta_j}{a+i\beta_j} e^{-i\beta_j T} \right] = \frac{a^2 \xi_j}{n_2} \frac{1}{i} \left(e^{i\beta_j T} - e^{-i\beta_j T} \right)$$

$$= \frac{1}{i} \frac{n_{1a}^2 + n_2(a^2+\beta_j^2)}{n_2} \left(e^{i\beta_j T} - e^{-i\beta_j T} \right).$$

Simplifying

$$e^{i\beta_j T} \left[\frac{n_1 a^2}{n_2} + a^2 + \beta_j^2 + (a^2+\beta_j^2) \left(\frac{a+i\beta_j}{a-i\beta_j} \right) \right] = e^{-i\beta_j T} \left[\frac{n_1 a^2}{n_2} + a^2 + \beta^2 + (a^2+\beta_j^2) \frac{a-i\beta_j}{a+i\beta_j} \right]$$

$$e^{2i\beta_j T} = \frac{\frac{n_1 a^2}{n_2} + a^2 + \beta_j^2 + (a-i\beta_j)^2}{\frac{n_1 a^2}{n_2} + a^2 + \beta_j^2 + (a+i\beta_j)^2} = \frac{\frac{n_1 a^2}{n_2} + 2a^2 - 2ia\beta_j}{\frac{n_1 a^2}{n_2} + 2a^2 + 2ia\beta_j} = e^{-2i \tan^{-1} \frac{\beta_j/a}{1+\frac{n_1}{2n_2}}} \,.$$

Hence, we have arrived at the condition

$$\frac{\beta_j/a}{1+\frac{n_1}{2n_2}} = -\tan \beta_j T \,, \quad j = 1, 2, \cdots, N-1. \tag{57}$$

which, in view of (48), is a condition imposed on the eigenvalues of γ. Since we assumed distinct eigenvalues for γ, we have to choose distinct roots of (57) (the case for degenerate eigenvalues needs further investigation).

Collecting the result for $\underset{\sim}{W}'$, we repeat here equations (38), (49), (51)

$$\underset{\sim}{W}' = \underset{\sim}{W}_a - e^{-i\omega T}\left[e^{i\omega T}\underset{\sim}{W}_a\right]_R , \tag{58}$$

$$\underset{\sim}{W}_a = -a(a+i\omega)\sum_1^{N-1}\frac{\xi_i}{n_i(\omega^2-\beta_i^2)}\epsilon_i \underset{\sim}{b} \tag{59}$$

$$\left[e^{i\omega T}\underset{\sim}{W}_a\right]_R = -\sum_1^{N-1} a\frac{\xi_i}{n_i}\epsilon_i\underset{\sim}{b}\left[\frac{a+i\beta_i}{\omega-\beta_i}\,\frac{e^{i\beta_i T}}{2\beta_i}+\frac{a-i\beta_i}{\omega+\beta_i}\,\frac{e^{-i\beta_i T}}{-2\beta_i}\right]. \tag{60}$$

If we deal with real solutions of (57), it is clear from the last equations that whether $\underset{\sim}{w}'$ can be made real depends only on whether we can find the appropriate constant vector $\underset{\sim}{b}$. This completes the construction in the transform domain.

Let us carry the results back to the time domain and, then, we shall summarize what has been found. We have in connection with $\underset{\sim}{W}_a$

$$IFT\left\{-\frac{(a+i\omega)}{\omega^2-\beta_i^2}\right\} = \frac{1}{2\pi i^2}\int_{-\infty}^{\infty}\frac{a+i\omega}{\omega^2-\beta_i^2}\,e^{i\omega T}d\omega = I. \tag{61}$$

We have, by Cauchy's theorem and the usual understanding of the meaning of an improper integral of the form I (that is, we are taking β_i real)

$$iI + \frac{1}{2\pi i}\left[\int_{C_1}+\int_{C_2}+\int_{C_3}\right]\frac{a+i\omega}{\omega^2-\beta_i^2}\,e^{i\omega t}d\omega = \sum_{UHP} res. = 0 \tag{62}$$

where the contains shown in Figure 2 are for $t > 0$

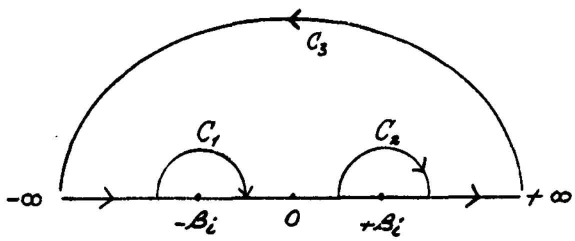

FIGURE 2

Thus, since the integral over C_3 vanishes

$$iI = \int_{C_1} + \int_{C_2} = \frac{1}{2}\left[\frac{a - i\beta_i}{-2\beta_i} e^{-i\beta_i t} + \frac{a + i\beta_i}{2\beta_i} e^{i\beta_i t}\right].$$

and so

$$I = \frac{\sqrt{a^2 + \beta_i^2}}{2\beta_i} \sin\left(\beta_i t + \tan^{-1}\frac{\beta_i}{a}\right) \qquad t > 0 \tag{63}$$

For $t < 0$, C_3 is taken in the LHP which results in

$$I = -\frac{\sqrt{a^2 + \beta_i^2}}{2\beta_i} \sin\left(\beta_i t + \tan^{-1}\frac{\beta_i}{a}\right)$$

so that

$$\underset{\sim}{W_a} = a\sum_1^{N-1} \frac{\xi_i \epsilon_i b}{n_2} \frac{\sqrt{a^2 + \beta_i^2}}{2\beta_i} \sin\left(\beta_i t + \tan^{-1}\frac{\beta_i}{a}\right). \tag{64}$$

Similarly, we find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{i\omega T}\underset{\sim}{W_a}\right]_R e^{i\omega t} d\omega - \frac{i}{2} a\sum_1^{N-1} \frac{\xi_i}{n_2} \epsilon_i b\left[\frac{a + i\beta_i}{2\beta_i} e^{i\beta_i(t+T)} + \frac{a - i\beta_i}{-2\beta_i} e^{-i\beta_i(t+T)}\right]. \tag{65}$$

Hence, the IFT of the second term of (58) is

$$-\operatorname{sgn}(t-T)\sum_1^{N-1} a\frac{\xi_i}{n_2} \epsilon_i \frac{b}{2\beta_i}\sqrt{a^2 + \beta_i^2} \sin\left(\beta_i t + \tan^{-1}\frac{\beta_i}{a}\right). \tag{66}$$

Combining, we have

$$\underset{\sim}{w}'\begin{cases}\sum_{1}^{N-1} a \frac{\xi_i}{n_2} \epsilon_i \frac{\underset{\sim}{b}'}{\beta_i} \sqrt{a^2+\beta_i^2} \sin\left(\beta_i t + \tan^{-1}\frac{\beta_i}{a}\right), \; 0 \leq t \leq T \\ 0, \text{ elsewhere}\end{cases} \tag{67}$$

Evaluating $\underset{\sim}{h}'(t) = w_0 * \underset{\sim}{w}'$ gives

$$\underset{\sim}{h}'(t) = \sum_{1}^{N-1} a^2 \frac{\xi_i \epsilon_i \underset{\sim}{b}'}{\beta_i n_2} \sin \beta_i t \,. \tag{68}$$

From Equation (25)

$$\underset{\sim}{f}(t) = \sum_{1}^{N-1} \frac{\xi_i \lambda^{-1} \nu_0}{\beta_i n_2} \epsilon_i \underset{\sim}{b}' a^2 \sin \beta_i (t-T) \,. \tag{69}$$

In summary, what we have obtained in the present section is that the solution vectors $\underset{\sim}{w}'$, $\underset{\sim}{f}$ are respectively linear combinations, Equations (67), (69), formed from the sets of functions

$$\sin\left(\beta_i t + \tan^{-1}\frac{\beta_i}{a}\right), \quad \sin \beta_i (t-T) \tag{70}$$

where the β_i are solutions of (57). The result is valid for general N. The difficulty in passing from the general form of the solution to the explicit solution i.e., choosing the appropriate β_i and $\underset{\sim}{b}'$, is that (48) has to be satisfied. The difficulty is greatly diminished in the case $N = 2$. For, then, (67) becomes simply

$$w_1' = w_1 - w_2 = A \sin\left(\beta_i t + \tan^{-1}\frac{\beta_i}{a}\right) \tag{67'}$$

where A is a constant. And, then, (68) and (69) become

$$h'(t) = \frac{Aa}{\sqrt{a^2+\beta_i^2}} \sin \beta_i t \tag{68'}$$

$$\underset{\sim}{f}(t) = \frac{Aa}{\sqrt{a^2+\beta_i^2}} \lambda^{-1} \nu_0 \sin \beta_i (t-T) \,. \tag{69'}$$

The last three equations are identical with (58), (62), (59) of Reference 1, while our present equation (48) is just the previous condition (54). Thus, in the case $N = 2$, we may show just as we did before that (48) is satisfied. But for $N > 2$, we have not as yet obtained an explicit solution.

In the next section, we present some orthogonality properties of the functions (70) together with some general properties of the basic vector equations (10), (11). Such properties are important for the completion of the solution for $N > 2$.

PROPERTIES RELATED TO THE SOLUTION OF THE MATRIC EQUATIONS

(a) Orthogonality Properties

We give, below, two important orthogonality properties of the sets of functions

$$h_i(t) = \sin \beta_i t \tag{71}$$

$$w_i(t) = \sin\left(\beta_i t + \tan^{-1} \frac{\beta_i}{a}\right), \tag{72}$$

where the β_i are solutions of the equation

$$\tan \beta_i T = -\frac{2n_2}{n_1 + 2n_2} \frac{\beta_i}{a} . \tag{73}$$

We may, without essential restriction, take the β_i positive and label them β_1, β_2, β_3, ... in increasing order of magnitude.

The results are

$$\int_0^T h_i(t)\, h_j(t)\, dt = 0, \quad i \neq j \tag{74}$$

$$\int_0^T \int_0^T w_i(u)\, w_j(v)\, \rho(u-v)\, du\, dv = 0, \quad i \neq j \tag{75}$$

where

$$\rho(\tau) = n_2\, \delta(\tau) + n_1 \frac{a}{2} e^{-a|\tau|} \tag{76}$$

Equation (76) is just the correlation function with which we have been dealing all along. It is conjectured, however, that the orthogonality properties corresponding to (74) and (75) will hold in more general circumstances. The proof of (74) is immediate on substituting (71) and using (73). The proof of (75) is also a straightforward calculation, only lengthy.

(b) Invariance Properties of the Matric, Vector Formulation

It may be verified from the definitions of ν , η in Reference 1 report that

$$\nu = \left(\frac{\partial P_c}{\partial m_{ij}}\right) , \quad \eta = -2\left(\frac{\partial P_c}{\partial \mu_{ij}}\right) . \tag{77}$$

By a notation of the form $\left(\frac{\partial P_c}{\partial m_{ij}}\right)$, we shall always mean the matrix whose ij element is the indicated quantity bearing the label ij .

The matric equations (10), (11) of Section 3 may, therefore, be written

$$\left(\frac{\partial P_c}{\partial m_{ij}}\right)(w_0 * \underset{\sim}{w})_{T-t} - \lambda \underset{\sim}{f}_t = \underset{\sim}{0} \tag{78}$$

$$t \in [0,T]$$

$$\widetilde{\left(\frac{\partial P_c}{\partial m_{ij}}\right)}(w_0 * \underset{\sim}{f})_{T-t} + 2\left(\frac{\partial P_c}{\partial \mu_{ij}}\right)\rho * \underset{\sim}{w}_t = \underset{\sim}{0} . \tag{79}$$

We consider, now, that a complete energy matrix has been specified. That is, that we are required to satisfy

$$\int_0^T \underset{\sim}{f}(t)\tilde{\underset{\sim}{f}}(t)\,dt = E \tag{80}$$

where the square matrix E is given. This is a deviation from the situation considered previously where E was a diagonal matrix, but the basic equations (10), (11), Section 3, remain unchanged, except that λ is no longer diagonal. Substituting (78) in (80) gives

$$\lambda = \left(\frac{\partial P_c}{\partial m_{ij}}\right)\tilde{m} E^{-1} \tag{81}$$

where $\tilde{m}$ is the transposed mean matrix. Substituting (81) in (78), we have

$$\left(\frac{\partial P_c}{\partial m_{ij}}\right)(w_0 * \underset{\sim}{w})_{T-t} - \left(\frac{\partial P_c}{\partial m_{ij}}\right)\tilde{m} E^{-1}\underset{\sim}{f}_t = \underset{\sim}{0} . \tag{82}$$

Consider the transformation

$$A\underset{\sim}{f} = \bar{\underset{\sim}{f}} , \quad \underset{\sim}{f} = \tilde{A}\bar{\underset{\sim}{f}} \tag{83}$$

$$A\underset{\sim}{w} = \bar{\underset{\sim}{w}} , \quad \underset{\sim}{w} = \tilde{A}\bar{\underset{\sim}{w}}$$

where A is an orthogonal matrix. Under this substitution, the matrices*
m, M, $\left(\frac{\partial P_c}{\partial m_{ij}}\right)$, $\left(\frac{\partial P_c}{\partial \mu_{ij}}\right)$ correspond to the new matrices $\bar{m}$, $\bar{M}$,
$\left(\frac{\partial P_c}{\partial \bar{m}_{ij}}\right)$, $\left(\frac{\partial P_c}{\partial \bar{\mu}_{ij}}\right)$. We find the relation between them as follows:

$$\bar{m} = w_0 * \bar{\underset{\sim}{f}} * \bar{\underset{\sim}{\tilde{w}}} = w_0 * A\underset{\sim}{f} * \underset{\sim}{\tilde{w}}\tilde{A} = A w_0 * \underset{\sim}{f} * \underset{\sim}{\tilde{w}}\tilde{A} = A m \tilde{A}. \tag{84}$$

And similarly,

$$\bar{M} = A M \tilde{A}. \tag{85}$$

For $\left(\frac{\partial P_c}{\partial \bar{m}_{ij}}\right)$, we have

$$\frac{\partial P_c}{\partial \bar{m}_{ij}} = \sum_{k,l} \frac{\partial P_c}{\partial m_{kl}} \frac{\partial m_{kl}}{\partial \bar{m}_{ij}}. \tag{86}$$

From (84)

$$\frac{\partial m_{kl}}{\partial \bar{m}_{ij}} = A_{ik} A_{jl}$$

and substitution in (86) gives

$$\left(\frac{\partial P_c}{\partial \bar{m}_{ij}}\right) = A \left(\frac{\partial P_c}{\partial m_{ij}}\right) \tilde{A}. \tag{85}$$

Similarly,

$$\left(\frac{\partial P_c}{\partial \bar{\mu}_{ij}}\right) = A \left(\frac{\partial P_c}{\partial \mu_{ij}}\right) \tilde{A}. \tag{86}$$

Collecting results, we substitute the following in (79), (82)

$$\underset{\sim}{f} = \tilde{A}\bar{\underset{\sim}{f}},\ \underset{\sim}{w} = \tilde{A}\bar{\underset{\sim}{w}},\ m = \tilde{A}\bar{m}A,\ M = \tilde{A}\bar{M}A$$

$$\left(\frac{\partial P_c}{\partial m_{ij}}\right) = \tilde{A}\left(\frac{\partial P_c}{\partial \bar{m}_{ij}}\right) A,\ \left(\frac{\partial P_c}{\partial \mu_{ij}}\right) = \tilde{A}\left(\frac{\partial P_c}{\partial \bar{\mu}_{ij}}\right) A.$$

This gives

$$\begin{aligned} &\left(\frac{\partial P_c}{\partial \bar{m}_{ij}}\right)(w_0 * \bar{\underset{\sim}{w}})_{T-t} - \left(\frac{\partial P_c}{\partial \bar{m}_{ij}}\right)\tilde{\bar{m}} E^{-1} \bar{\underset{\sim}{f}}_t = \underset{\sim}{0} \\ &\left(\frac{\tilde{\partial} P_c}{\partial \bar{m}_{ij}}\right)(w_0 * \bar{\underset{\sim}{f}})_{T-t} + 2\left(\frac{\partial P_c}{\partial \bar{\mu}_{ij}}\right)\rho * \bar{\underset{\sim}{w}}_t = \underset{\sim}{0} \end{aligned} \tag{87}$$

* We have used the definitions of m, M given by Equations (2), (6) of Reference 1.

where $\bar{E} = AE\tilde{A}$.

Thus, if $\underset{\sim}{f}$, $\underset{\sim}{w}$ solve the system when E is given, then $\bar{\underset{\sim}{f}} = A\underset{\sim}{f}$, $\bar{\underset{\sim}{w}} = A\underset{\sim}{w}$ solve the system when $\bar{E} = AE\tilde{A}$ is given.

We consider next the substitution

$$\begin{aligned} \bar{\underset{\sim}{w}} &= k\underset{\sim}{w}, \quad \bar{\underset{\sim}{f}} = \underset{\sim}{f} \\ \underset{\sim}{w} &= \frac{1}{k}\bar{\underset{\sim}{w}} \end{aligned} \quad , k \neq 0 . \tag{88}$$

We have at once

$$\begin{aligned} \bar{m} &= km \\ \bar{M} &= k^2 M \\ \left(\frac{\partial P_c}{\partial \bar{m}_{ij}}\right) &= \frac{1}{k}\left(\frac{\partial P_c}{\partial m_{ij}}\right) \\ \left(\frac{\partial P_c}{\partial \bar{\mu}_{ij}}\right) &= \frac{1}{k^2}\left(\frac{\partial P_c}{\partial \mu_{ij}}\right) . \end{aligned} \tag{89}$$

Substituting in (79) and (82), the equations are unaltered and we conclude that if $\underset{\sim}{f}$, $\underset{\sim}{w}$ solve the system, so does $\underset{\sim}{f}$, $k\underset{\sim}{w}$. Moreover, we deduce easily from the representation of P_i given in Equation (91) of the following section that

$$P_c(m, M) = P_c(km, k^2 M)$$

where it is recalled that

$$P_c = \sum p_i P_i .$$

Hence, $\underset{\sim}{w}$ is arbitrary to a multiplicative constant and different constants give the same P_c .

AN ALTERNATE REPRESENTATION OF CONDITIONAL PROBABILITIES OF CORRECT DECISION

A contour integral representation of the conditional probabilities of correct decision, P_i , was given in Equation (14) of Reference 1. It was obtained by use of the characteristic function of the normal distribution and was valid for an arbitrary number N of signals and weighting functions. We shall obtain directly here an alternate representation which, in some respects, is more convenient than the previous one. We shall discuss it relative to the case $N = 3$, but it will be evident that the essential arguments are valid for any N .

For $N = 3$, the conditional probabilities of correct decision are

$$\begin{aligned} P_1 &= P(x_1 > x_2,\ x_1 > x_3 \mid f_1 \text{ sent}) \\ P_2 &= P(x_2 > x_1,\ x_2 > x_3 \mid f_2 \text{ sent}) \\ P_3 &= P(x_3 > x_1,\ x_3 > x_2 \mid f_3 \text{ sent}) \ . \end{aligned} \tag{90}$$

For the first of these, we have

$$P_1 = \frac{|M|^{-1/2}}{(2\pi)^{3/2}} \int_S e^{-1/2(\widetilde{x-m_1})M^{-1}(x-m_1)}\, dx_1\, dx_2\, dx_3 \tag{91}$$

where m_1 is the moment vector (column matrix) when f_1 is sent

M is the moment matrix

and S is the subset of E^3 which is bounded by the planes

$$x_1 - x_2 = 0\ , \quad x_1 - x_3 = 0$$

and contains the point m_1. (For convenience, we have dispensed with the tilde used elsewhere to distinguish column matrices or vectors.) By a preliminary translation, we get

$$P_1 = \frac{|M|^{-1/2}}{(2\pi)^{3/2}} \int_{S'} e^{-\frac{1}{2}\tilde{x}M^{-1}x}\, dx_1\, dx_2\, dx_3$$

where S' now contains the origins and is bounded by the planes

$$\begin{aligned} x_1 - x_2 + m_{11} - m_{12} &= 0 \\ x_1 - x_3 + m_{11} - m_{13} &= 0 \end{aligned} \tag{92}$$

Since M is positive definite, we may make the substitution (Reference 2)

$$x = M^{1/2} x' \tag{93}$$

and obtain

$$P_1 = \frac{1}{(2\pi)^{3/2}} \int_{S''} e^{-1/2 \sum_1^3 x_i^2}\, dx_1\, dx_2\, dx_3 \tag{94}$$

where S'' is the image of S' under the transformation $M^{1/2}$. It is convenient to write the planes (92) in the more general notation

$$\tilde{a}x + m_{11} - m_{12} = 0$$
$$\tilde{b}x + m_{11} - m_{13} = 0 \tag{95}$$

where, in the present case,

$$a = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{96}$$

Under the linear substitution (4) the planes (6) transform into the planes

$$\tilde{\bar{a}}x + m_{11} - m_{12} = 0$$
$$\tilde{\bar{b}}x + m_{11} - m_{13} = 0 \tag{97}$$

where

$$\tilde{\bar{a}} = \tilde{a}M^{1/2}, \quad \tilde{\bar{b}} = \tilde{b}M^{1/2}.$$

Since $M^{1/2}$ is symmetric,

$$\bar{a} = M^{1/2}a, \quad \bar{b} = M^{1/2}b. \tag{98}$$

There is an orthogonal transformation T which carries the line of intersection of the planes (97) parallel to the x_3 axis (in fact, there are an infinity of such transformations, any one of which will do for our purposes). Introducing in (94) the change of variable $y = Tx$ gives, since $|det\, T| = 1$,

$$P_1 = \frac{1}{(2\pi)^{3/2}} \int_{S'''} e^{-1/2 \sum_{i=1}^{3} y_i^2} dy_1\, dy_2\, dy_3 \tag{99}$$

where S''' is a cylinder set parallel to x_3. Hence, we may integrate out y_3 and obtain

$$P_1 = \frac{1}{2\pi} \int_{S_2} e^{-1/2 (y_1^2 + y_2^2)} dy_1\, dy_2 \tag{100}$$

where S_2 is a subset of the plane which may be described as follows. The distances from the origin to the planes (97) and the angle between them are given by

$$d_1 = \frac{m_{11} - m_{12}}{(\tilde{\bar{a}}\bar{a})^{1/2}}, \quad d_2 \frac{m_{11} - m_{13}}{(\tilde{\bar{b}}\bar{b})^{1/2}}, \quad \cos\theta = \frac{\tilde{\bar{a}}\bar{b}}{(\tilde{\bar{a}}\bar{a})^{1/2}(\tilde{\bar{b}}\bar{b})^{1/2}}. \tag{101}$$

These quantities are, of course, invariant under orthogonal transformation and, therefore, (101) gives the distances to, and the angle between, the lines which bound S_2 . Using (98) in (101) gives

$$d_1 = \frac{m_{11} - m_{12}}{(\tilde{a} M a)^{1/2}} , \quad d_2 = \frac{m_{11} - m_{13}}{(\tilde{b} M b)^{1/2}} , \quad \cos\theta = \frac{\tilde{a} M b}{(\tilde{a} M a)^{1/2} (\tilde{b} M b)^{1/2}} \tag{102}$$

so that the parameters which describe S_2 , and therefore P_1 , are expressed simply in terms of linear and bilinear forms with matrix M . Thus, with the use of (96), we have finally

$$d_1 = \frac{m_{11} - m_{12}}{(\mu_{11} - 2\mu_{12} + \mu_{22})^{1/2}} , \quad d_2 = \frac{m_{11} - m_{13}}{(\mu_{11} - 2\mu_{13} + \mu_{33})^{1/2}} \tag{103}$$

$$\cos\theta = \frac{\mu_{11} - \mu_{13} - \mu_{21} + \mu_{23}}{(\mu_{11} - 2\mu_{12} + \mu_{22})^{1/2} (\mu_{11} - 2\mu_{13} + \mu_{33})^{1/2}}$$

where the μ_{ij} are the elements of M . Similarly, for P_2 and P_3 , we have

$$d_1^{(2)} = \frac{m_{22} - m_{21}}{(\mu_{11} - 2\mu_{12} + \mu_{22})^{1/2}} , \quad d_2^{(2)} = \frac{m_{22} - m_{23}}{(\mu_{22} - 2\mu_{23} + \mu_{33})^{1/2}} \tag{104}$$

$$d_1^{(3)} = \frac{m_{33} - m_{31}}{(\mu_{33} - 2\mu_{13} + \mu_{11})^{1/2}} , \quad d_2^{(3)} = \frac{m_{33} - m_{32}}{(\mu_{33} - 2\mu_{23} + \mu_{22})^{1/2}}$$

$$\cos\theta^{(2)} = \frac{\mu_{22} - \mu_{23} - \mu_{21} + \mu_{13}}{(\mu_{22} - 2\mu_{23} + \mu_{33})^{1/2} (\mu_{11} - 2\mu_{12} + \mu_{22})^{1/2}} , \quad \cos\theta^{(3)} = \frac{\mu_{33} - \mu_{31} - \mu_{32} + \mu_{21}}{(\mu_{33} - 2\mu_{13} + \mu_{11})^{1/2} (\mu_{33} - 2\mu_{23} + \mu_{22})^{1/2}}$$

A convenient formula for computation of the P_i's for N = 3 has been obtained from (100), (103), (104) while further work is needed to obtain a computational form for larger N .

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VIII

RECOMMENDATIONS FOR FURTHER INVESTIGATIONS

1. Maximum Likelihood Reception of Frequency Modulated Signals

This investigation should be extended to include operation near and below the FM threshold. If the results obtained are superior to those obtainable by other means, such as FMFB, then means of implementing maximum likelihood FM receivers should be investigated.

2. Threshold Performance in FM Systems

In Chapter III, the threshold characteristic of an FM receiver consisting of a limiter-discriminator followed by a minimum mean-square-error postdetection filter (Wiener filter) was investigated. The modulation function was assumed to be a gaussian random variable which made the determination of the required IF receiver bandwidth rather difficult and somewhat arbitrary.

It would be desirable to extend this work to include other forms of modulating functions, the statistics of which would be closer to those of signal functions encountered in practice. It is suggested that a similar analysis be carried through for a band-limited modulation function having a uniform distribution of amplitude over a given range. This would be more representative of practical situations and, also, would lead to a better defined bandwidth of the transmitted signal. This would also allow the results of the FM analysis to be compared with the PCM analysis in Chapter V.

The work in Chapter III considered only the effects of the additive, white, gaussian noise source in determining the output signal-to-noise ratio. The noise power and, hence, the position of the threshold is quite dependent on the IF bandwidth selected. Thus, from the standpoint of reducing noise (and, hence, threshold), it would be desirable to reduce the bandwidth; however, any reduction in bandwidth is accompanied by increasing distortion due to truncation of the IF signal spectrum. This work should be extended to establish, quantitatively, the most desirable IF bandwidth in order to optimize over-all performance when considering both the additive noise and signal distortion effects.

3. Use of Information Theory to Bound the Performance of Communications Systems

The bound derived in this report is on the ratio of signal entropy power to mean square error in terms of channel capacity. This result has two serious shortcomings. First, we do not know how to attain the bound, but anticipate that a close approach to the bound would entail a very lengthy coding procedure. Secondly, the practical significance of a bound on the ratio of signal entropy power to mean square error is not immediately apparent. With regard to the first point, one may be able to obtain bounds for codes of finite complexity by proceeding in a manner similar to that outlined in Chapter V.

4. Investigation of Transmission of Analog Data Over a Digital Channel

Further consideration should be given to the selection of the performance criteria (S/N, MSE, etc.) in terms of the system application.

A comparison should be made with conventional analog systems (e. g., FM and FMFB) to establish the relative merits of analog and analog-digital systems as a function of channel parameters, bandwidth-expansion factors, required average power, etc.

The investigation of the effects of different error distributions should be continued. The distribution of the digital errors may be manipulated in several ways; for instance, in a PCM system, different energies may be assigned (by varying the duration or amplitude) to the various bits of a code word. The ability to alter the error probabilities may be exploited in a manner akin to predistortion of analog signals, such as pre-emphasis in FM systems. Theoretical bounds for such systems with nonuniform error probabilities need to be developed.

The system performance when an analog signal is transmitted by a digital system over a fading channel should be investigated, and a comparison should be made with direct analog methods operating over an equivalent channel.

5. Optimization of Digital Communications Systems Operating Over Dispersive Channels

By system optimization is meant the simultaneous specification of the transmitted waveforms and the receiver, so as to obtain the minimum probability of error under the given restraints.

The entire solution to this problem has not yet been obtained even for the simplest cases considered, except for the case $N = 2$. The present state of affairs can perhaps best be summed up by stating that, for a known channel transfer function and noise statistics,

a. given the set of transmitted signals, the best receiver configuration can be determined, or

b. given the receiver configuration, the best set of signals to transmit can be determined.

Although we have expended considerable effort at attempts to obtain simultaneous optimization, we have so far not been successful. This, therefore, remains an open problem. It is noted that, in the radar field, a great amount of effort recently has been devoted to signal synthesis. Many valuable results have been obtained, although no real optimum has been found. Therefore, it seems reasonable to expend further effort at improving system performance even if the optimum remains elusive for the present.

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